

Regularization of odd-dimensional AdS gravity: Kounterterms

Rodrigo Olea

*Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física,
Instituto Superior Técnico, Universidade Técnica de Lisboa,
Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
E-mail: rolea@fisica.ist.utl.pt*

ABSTRACT: As an alternative to the standard Dirichlet counterterms prescription, I introduce the concept of *Kounterterms* as the boundary terms with explicit dependence on the extrinsic curvature K_{ij} that regularize the AdS gravity action. A suitable choice of the boundary conditions — compatible with any asymptotically AdS (AAAdS) spacetime — ensures a finite action principle for all odd dimensions. Background-independent conserved quantities are obtained as Noether charges associated to asymptotic symmetries and their general expression appears naturally split in two parts. The first one gives the correct mass and angular momentum for AAAdS black holes and vanishes identically for globally AdS spacetimes. Thus, the second part is a covariant formula for the vacuum energy in AAAdS spacetimes and reproduces the results obtained by the Dirichlet counterterms method in a number of cases. It is also shown that this Kounterterms series regularizes the Euclidean action and recovers the correct black hole thermodynamics in odd dimensions.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Classical Theories of Gravity.

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1. Introduction

In the context of AdS/CFT correspondence [1], Witten sketched the program to regularize the action for AdS spacetimes [2], which was carried out in detail by Hennigson and Skenderis in ref. [3].

This procedure, known as holographic renormalization, considers a generic form of the metric for an asymptotically AdS (AAdS) spacetime

$$ds^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{g_{ij}(\rho, x)}{\rho} dx^i dx^j \quad (1.1)$$

where ρ is the radial coordinate of a manifold M , whose boundary is located at $\rho = 0$, and ℓ is the AdS radius. This coordinates choice is suitable to describe the conformal structure of the boundary, whose metric $g_{ij}(\rho, x)$ accepts a regular expansion [4]

$$g_{ij}(\rho, x) = g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \dots \tag{1.2}$$

where $g_{(0)ij}$ is a given initial value for the metric.

Solving the Einstein equations in this frame reconstructs the spacetime from the boundary data, determining the coefficients $g_{(k)}$ as covariant functionals in the boundary metric $g_{(0)}$ (that contain k derivatives of x^i). Then, in order to preserve general covariance at the boundary, it is necessary to invert the series to express all the quantities as functions of the boundary metric $h_{ij} = g_{ij}/\rho$ [5].

In this way, the counterterms method proposes a regularization scheme that consists in the addition to the Dirichlet action of local functionals of the boundary metric h_{ij} , the intrinsic curvature R_{ij}^{kl} of the boundary and covariant derivatives of the boundary Riemann $\nabla_m R_{ij}^{kl}$. In $D = d + 1$ dimensions the regularized action reads

$$I = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-\mathcal{G}} (\hat{R} - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-h} K + \int_{\partial M} d^d x \mathcal{B}_{ct}(h, R, \nabla R). \tag{1.3}$$

In the above action, hatted curvatures refer to $(d + 1)$ -dimensional ones, the cosmological constant is $\Lambda = -d(d - 1)/2\ell^2$ and $K = K_{ij}h^{ij}$ is the trace of the extrinsic curvature.

The Gibbons-Hawking term ensures a well-posed variational principle for a Dirichlet boundary condition on the metric h_{ij} [6], that is still valid in presence of the Dirichlet counterterms series because its functional variation is expressed as $\delta\mathcal{B}_{ct} = (\delta\mathcal{B}_{ct}/\delta h_{ij})\delta h_{ij}$.

As a consequence, a regularized stress tensor for AdS spacetimes is obtained [7, 8] using the Brown-York quasilocal energy-momentum tensor definition [9], without reference to any background solution.

A novel feature of this approach is the appearance — in $D = 2n + 1$ dimensions — of a vacuum energy for AdS spacetime, that is clearly unobservable in background-dependent methods. In five dimensions, the matching between this vacuum energy and the Casimir energy induced by a precise boundary CFT ($\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$) is one of the best known examples of the AdS/CFT correspondence [7].

Despite the fact this regularization procedure provides a systematic way to construct the Dirichlet counterterms series, in practice, the number of possible counterterms increase drastically with the dimension. Even for a given dimension, the finiteness of the conserved charges for a more complex solution would require a significant addition of counterterms respect to the same problem, for instance, in Schwarzschild-AdS black hole. Moreover, these extra terms do not seem to obey any particular pattern [10].

In recent papers [11, 12], the problem of Dirichlet counterterms in AdS gravity has been reformulated as an initial-value problem for the extrinsic curvature. This results in a simpler algorithm to obtain the series \mathcal{B}_{ct} but, however, the full series for an arbitrary dimension is still unknown.

Furthermore, in the counterterms method is not clear where the vacuum energy is coming from, i.e., which boundary terms are responsible for the shifting of the zero-point energy of AdS or whether it can be obtained from a general covariant formula for any AAdS spacetime.

Therefore, a natural question arises: Is there another counterterms series that also regularizes the Einstein-Hilbert action with negative cosmological constant, and whose expression can be worked out in any dimension?

Certainly the answer to that problem implies the departure from an standard action principle based on a Dirichlet boundary condition on the metric h_{ij} .

Such construction may at first sound too ambitious. However, we have evidence coming from even-dimensional AdS gravity, where considering a boundary condition for the spacetime curvature

$$\hat{R}^{\mu\nu}_{\alpha\beta} + \frac{1}{\ell^2} \delta^{\mu\nu}_{[\alpha\beta]} = 0 \tag{1.4}$$

on ∂M has been a good alternative to produce a finite action principle [13, 14]. In that case, the gravity action is supplemented by the Euler term \mathcal{E}_{2n} with a coupling constant fixed demanding this generic asymptotic condition.

In $D = 2n$ dimensions, the Euler theorem states that the Euler term \mathcal{E}_{2n} is equivalent to the boundary term B_{2n-1} (the n -th Chern form) up to the Euler characteristic χ_{2n} , a topological number for the manifold M . From a dynamic point of view, χ_{2n} is just an integration constant and then, the formula for the conserved charges is the same if we supplement the Einstein-Hilbert-AdS action either with the boundary term B_{2n-1} or the Euler (bulk) term \mathcal{E}_{2n} [15]. In that sense, the regularizing effect of the Dirichlet counterterm series can be replaced by the Euler term in even dimensions. The clear advantage of this procedure is that we do not need to perform any particular expansion of the metric, nor solving the asymptotic equations in that frame to find the coefficients $g_{(k)ij} = g_{(k)ij}(g_{(0)})$, nor inverting the series to express all the quantities as covariant functionals of the boundary metric h_{ij} . This procedure contrasts with the simplicity of fixing a single global factor in \mathcal{E}_{2n} , given by the asymptotic condition (1.4). In other words, considering the Euler term as a single entity, its coupling constant comes from fixing the leading order in the curvature, because in the expansion (1.1), (1.2), the asymptotic Riemann reads

$$\hat{R}^{k\rho}_{ij}(\mathcal{G}) = O(\rho^2), \tag{1.5}$$

$$\hat{R}^{k\rho}_{i\rho}(\mathcal{G}) = -\frac{1}{\ell^2} \delta_i^k + O(\rho^2), \tag{1.6}$$

$$\hat{R}^{kl}_{ij}(\mathcal{G}) = -\frac{1}{\ell^2} \delta^{[kl]}_{[ij]} + \rho \left(R^{kl}_{ij}(g_{(0)}) + \frac{1}{\ell^2} \left(\delta^k_{[i} g^l_{(1)j]} + g^k_{(1)[i} \delta^l_{j]} \right) \right) + O(\rho^2), \tag{1.7}$$

where $g^i_{(1)j} = g^{il}_{(0)} g_{(1)lj}$.

In sum, what is remarkable in this approach is the fact that we have a closed expression for the boundary term B_{2n-1} in all even dimensions and also, a deep connection between the regularizing boundary terms and topological invariants.

The same approach cannot be applied to odd-dimensional AdS gravity, because there are no topological invariants of the Euler class in $D = 2n + 1$, what makes gravity in even

and odd dimensions quite different. This is not so surprising, because also in standard holographic renormalization there appear technical differences respect the even-dimensional case: the expansion (1.2) requires a $\log \rho$ term at order ρ^n to be consistent with the equations of motion, the existence of Weyl anomaly, the appearance of a vacuum energy for AdS space, etc.

Following a different strategy, a mechanism to regularize the AdS gravity action in odd dimensions was proposed in [16]. A boundary condition for the extrinsic curvature K_i^j in AAdS spaces is the key assumption that leads to a well-posed action principle for a given boundary term B_{2n} . This asymptotic condition was motivated by a similar construction in Chern-Simons-AdS gravity [17].

Indeed, in the three-dimensional case, where Einstein-Hilbert-AdS gravity action is a Chern-Simons form for the group $SO(2,2)$, this prescription regularizes the Euclidean action and the Noether charges with a single boundary term that is one half the Gibbons-Hawking term. It can be proved that a suitable expansion reduces the problem to the Dirichlet formulation plus a topological invariant of the boundary metric [18].

At this point, as an alternative to the standard counterterms approach, we introduce the concept of *Kounterterms* as the boundary terms that regularize the AdS gravity action, that posses explicit dependence on the extrinsic curvature K_{ij} and whose construction is based on boundary conditions compatible with the Fefferman-Graham form of the metric (1.2).

In this article, we show the general tensorial form the Kounterterms adopt in the odd-dimensional case.

2. Kounterterms

We consider the Einstein-Hilbert action with negative cosmological constant, with the addition of a boundary term B_d

$$I = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-\mathcal{G}} \left(\hat{R} - 2\Lambda \right) + c_d \int_{\partial M} d^d x B_d \tag{2.1}$$

instead of the standard Gibbons-Hawking term plus the counterterms series. Here, c_d is a coupling constant that will be determined by an appropriate variational principle.

We consider a radial foliation for the spacetime (Gaussian normal coordinates)

$$ds^2 = N^2(\rho) d\rho^2 + h_{ij}(\rho, x) dx^i dx^j, \tag{2.2}$$

but we do not assume any particular expansion of the boundary metric as in eq. (1.1), (1.2). In this frame, the expression for the extrinsic curvature adopts a simple form

$$K_{ij} = -\frac{1}{2N} \partial_\rho h_{ij}. \tag{2.3}$$

2.1 Kounterterms and the Euler theorem

The Kounterterms series differs from the one obtained from the Dirichlet regularization, even in a simple case as it is $D = 4$, where it is given by the boundary term [15]

$$B_3 = 2\sqrt{-h} \delta_{[j_1 j_2 j_3]}^{[i_1 i_2 i_3]} K_{i_1}^{j_1} \left(R_{i_2 i_3}^{j_2 j_3}(h) - \frac{2}{3} K_{i_2}^{j_2} K_{i_3}^{j_3} \right), \tag{2.4}$$

with a coupling constant $c_3 = \ell^2/(64\pi G)$. In the above formula, $R_{kl}^{ij}(h)$ stands for the intrinsic curvature of the boundary metric.

It is clear from the expanded form

$$B_3 = 4\sqrt{-h} \left[-\frac{2}{3}K_j^i K_k^j K_i^k + K \left(K_j^i K_i^j - \frac{1}{3}K^2 \right) - 2 \left(R_j^i - \frac{1}{2}\delta_j^i R \right) K_i^j \right], \quad (2.5)$$

that B_3 does not contain any term proportional to $\sqrt{-h}K$ (Gibbons-Hawking term), making evident that it is not derived from a Dirichlet action principle. Notice that the dimensional continuation of (2.5) is required to define the Dirichlet problem for Einstein-Gauss-Bonnet gravity in $D \geq 5$, because it is the generalization of the Gibbons-Hawking term for the quadratic terms in the curvature in the Gauss-Bonnet density [19, 20].

The expression (2.4) possesses the additional Lorentz symmetry in the tangent space that becomes manifest when it is expressed in terms of the second fundamental form (SFF)

$$\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}, \quad (2.6)$$

defined as the difference between the dynamic spin connection $\omega^{AB} = \omega_\mu^{AB} dx^\mu$ and a reference one $\bar{\omega}^{AB}$.

The indices of the tangent space run in the set $A, B = \{0, 1, \dots, D-1\}$. In the metric formulation of gravity, the spin connection is determined in terms of the vielbein $e^A = e_\mu^A dx^\mu$ ($\mathcal{G}_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B$) as

$$\omega_\mu^{AB} = -e^{B\nu} \nabla_\mu e_\nu^A. \quad (2.7)$$

For the radial foliation (2.2), the natural splitting for the orthonormal basis is

$$e^1 = N d\rho, \quad (2.8)$$

$$e^a = e_i^a dx^i, \quad (2.9)$$

where the indices set $A = \{1, a\}$ and the boundary metric is $h_{ij} = \eta_{ab} e_i^a e_j^b$.

An adequate choice of the reference connection $\bar{\omega}^{AB}$, as obtained from a cobordant product metric

$$ds^2 = \bar{N}^2(\rho) d\rho^2 + \bar{h}_{ij}(x) dx^i dx^j \quad (2.10)$$

that matches the dynamic one only on the boundary $\rho = \rho_0$, ($\bar{h}_{ij}(x) = h_{ij}(\rho_0, x)$) leads to a SFF on the boundary as [21–23, 19]

$$\theta^{1a} = K_i^a dx^i, \quad \theta^{ab} = 0, \quad (2.11)$$

in terms of the extrinsic curvature $K_i^a = e_j^a K_i^j$. We stress that the spin connection $\bar{\omega}^{AB}$ is just introduced on ∂M to restore Lorentz covariance in the boundary term but it is not related to any background-subtraction procedure, where the background needs to be a solution of the bulk field equations.

Therefore, the fully Lorentz-covariant expression for the Kounterterms B_3 in terms of the Levi-Civita tensor is¹

$$B_3 = 2\varepsilon_{ABCD} \theta^{AB} \left(R^{CD} + \frac{1}{3} \theta_F^C \theta^{FD} \right). \quad (2.12)$$

¹The wedge product \wedge between the differential forms is omitted throughout the paper.

In four-dimensional manifolds without boundary, the integration of the Euler-Gauss-Bonnet term

$$\mathcal{E}_4 = \varepsilon_{ABCD} \hat{R}^{AB} \hat{R}^{CD} = -d^4x \sqrt{-\mathcal{G}} \left(\hat{R}_{\mu\nu\alpha\beta} \hat{R}^{\mu\nu\alpha\beta} - 4\hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{R}^2 \right)$$

is simply proportional to the Euler characteristic $\chi(M_4)$. When a boundary is introduced, the Euler theorem states that there is a correction due to the boundary

$$\int_{M_4} \varepsilon_{ABCD} \hat{R}^{AB} \hat{R}^{CD} = 2(4\pi)^2 \chi(M_4) + 2 \int_{\partial M_4} \varepsilon_{ABCD} \theta^{AB} \left(R^{CD} + \frac{1}{3} (\theta^2)^{CD} \right), \quad (2.13)$$

given exactly by the expression (2.12) that is known as the second Chern form. In fact, using the general formalism reviewed in appendix A, the boundary term can be seen as a transgression form for the Lorentz group $SO(3, 1)$.

In higher even-dimensional AdS gravity, the regularization of the conserved quantities was achieved in the ref. [14] by the addition of the Euler term \mathcal{E}_{2n} . This term is no longer equivalent to the Gauss-Bonnet term, because it is the maximal Lovelock form in that dimension. This construction is locally equivalent, by virtue of the Euler theorem, to the n -th Chern form

$$B_{2n-1} = n \int_0^1 dt \varepsilon_{A_1 \dots A_{2n}} \theta^{A_1 A_2} (R^{A_3 A_4} + t^2 \theta_F^{A_3} \theta^{F A_4}) \times \dots \times (R^{A_{2n-1} A_{2n}} + t^2 \theta_F^{A_{2n-1}} \theta^{F A_{2n}}) \quad (2.14)$$

in terms of the continuous parameter t . This parametrization is useful not only to write down a compact formula for the boundary term but to generate the relative coefficients of the binomial expansion, as well. This fact has a deep geometrical origin as an explicit realization of the Cartan homotopy operator, that permits to obtain the explicit form of a boundary term whose exterior derivative is the difference of two invariant polynomials for a given Lie group (See appendix A). It can be shown that the above boundary term also cancels the divergences from radial infinity in the evaluation of the bulk Euclidean action [15].

2.2 Kounterterms and Chern-Simons forms

Remarkably, the regularization prescription given by the maximal Chern-form in even dimensions works equally well for Einstein-Gauss-Bonnet gravity, where the coupling constant carried by B_{2n-1} takes a different value respect to the same problem in Einstein-Hilbert [24]. The same situation is found in a generic even-dimensional Lovelock-AdS theory, where the Kounterterms series (2.14) preserves its form, but again its coupling constant changes accordingly [25]. This fact strongly suggests the universality of the boundary terms that regularize the action for a set of inequivalent gravity theories (at least, the ones that are Lovelock-type).

On the other hand, a finite action principle was set for Chern-Simons-AdS gravity, a particular Lovelock theory in odd-dimensions that possesses a unique cosmological constant, contains higher powers in the curvature and can be obtained from a Chern-Simons

form for the AdS group connection. The guiding line to derive the correct form of the boundary terms is restoring gauge invariance by the use of transgression forms.

As we can see from appendix A, the expression for the Kounterterms in Chern-Simons-AdS gravity is given by a double integral in the parameters $t, s \in [0, 1]$

$$\begin{aligned}
 B_{2n} = & n \int_0^1 dt \int_0^t ds \varepsilon_{A_1 \dots A_{2n+1}} \theta^{A_1 A_2} e^{A_3} \left(R^{A_4 A_5} + t^2 \theta_F^{A_4} \theta^{F A_5} + \frac{s^2}{\ell^2} e^{A_4} e^{A_5} \right) \times \dots \\
 & \dots \times \left(R^{A_{2n} A_{2n+1}} + t^2 \theta_F^{A_{2n}} \theta^{F A_{2n+1}} + \frac{s^2}{\ell^2} e^{A_{2n}} e^{A_{2n+1}} \right). \tag{2.15}
 \end{aligned}$$

In the spirit of the alternative regularization of AdS gravity in even dimensions, we will assume universality of the form of the regularizing boundary terms in $D = 2n + 1$ dimensions. In fact, as we shall explicitly demonstrate below, the boundary term (2.15) also leads to a finite, well-defined action principle in odd-dimensional Einstein-Hilbert for a suitable choice of its coupling constant c_{2n} .

The relations (2.11) leave a residual Lorentz symmetry on ∂M and then, the boundary term can also be written as

$$\begin{aligned}
 B_{2n} = & -2n \int_0^1 dt \int_0^t ds \varepsilon_{a_1 \dots a_{2n}} K^{a_1} e^{a_2} \left(R^{a_3 a_4} - t^2 K^{a_3} K^{a_4} + \frac{s^2}{\ell^2} e^{a_3} e^{a_4} \right) \times \dots \\
 & \dots \times \left(R^{a_{2n-1} a_{2n}} - t^2 K^{a_{2n-1}} K^{a_{2n}} + \frac{s^2}{\ell^2} e^{a_{2n-1}} e^{a_{2n}} \right), \tag{2.16}
 \end{aligned}$$

where R^{ab} is the boundary 2-form curvature, related to the intrinsic curvature by $R^{ab} = \frac{1}{2} R_{ij}^{kl} e_k^a e_l^b dx^i \wedge dx^j$.

The tensorial form of the Kounterterms can be worked out projecting all the quantities in the boundary indices (see appendix B)

$$\begin{aligned}
 B_{2n} = & 2n \sqrt{-h} \int_0^1 dt \int_0^t ds \delta_{[j_1 \dots j_{2n-1}]^{[i_1 \dots i_{2n-1}]} K_{i_1}^{j_1} \left(\frac{1}{2} R_{i_2 i_3}^{j_2 j_3} - t^2 K_{i_2}^{j_2} K_{i_3}^{j_3} + \frac{s^2}{\ell^2} \delta_{i_2}^{j_2} \delta_{i_3}^{j_3} \right) \times \dots \\
 & \dots \times \left(\frac{1}{2} R_{i_{2n-2} i_{2n-1}}^{j_{2n-2} j_{2n-1}} - t^2 K_{i_{2n-2}}^{j_{2n-2}} K_{i_{2n-1}}^{j_{2n-1}} + \frac{s^2}{\ell^2} \delta_{i_{2n-2}}^{j_{2n-2}} \delta_{i_{2n-1}}^{j_{2n-1}} \right), \tag{2.17}
 \end{aligned}$$

In the language of differential forms, the gravitational action (2.1) can be written in terms of the local orthonormal frame $e^A = e_\mu^A dx^\mu$ and the 2-form Lorentz curvature $\hat{R}^{AB} = \frac{1}{2} \hat{R}_{\mu\nu}^{AB} dx^\mu \wedge dx^\nu$ (constructed up from the spin connection as $\hat{R}^{AB} = d\omega^{AB} + \omega_C^A \omega^{CB}$) as

$$I = \kappa_D \int_M \varepsilon_{A_1 \dots A_D} \left(\hat{R}^{A_1 A_2} + \frac{(D-2)}{D\ell^2} e^{A_1} e^{A_2} \right) e^{A_3} \dots e^{A_D} + c_d \int_{\partial M} B_d. \tag{2.18}$$

where the constant $\kappa_D = (16\pi G(D-2)!)^{-1}$ and with the boundary term B_d given by (2.14) and (2.15) for even and odd dimensions, respectively. The Lorentz curvature is related to the spacetime Riemann tensor by $\hat{R}^{AB} = \frac{1}{2} \hat{R}_{\mu\nu}^{\alpha\beta} e_\alpha^A e_\beta^B dx^\mu \wedge dx^\nu$.

An arbitrary variation of the above action produces the Einstein equations plus a surface term Θ

$$\delta I = \int_M E_A \delta e^A + d\Theta, \tag{2.19}$$

where $E_A \delta e^A$ is the Einstein equation,

$$E_A \delta e^A = \frac{1}{16\pi G(D-3)!} \varepsilon_{AA_2 \dots A_D} \delta e^A \left(\hat{R}^{A_2 A_3} + \frac{1}{\ell^2} e^{A_2} e^{A_3} \right) e^{A_4} \dots e^{A_D}, \quad (2.20)$$

$$= \frac{1}{16\pi G} \sqrt{-\mathcal{G}} \left(\hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \mathcal{G}_{\mu\nu} - \Lambda \mathcal{G}_{\mu\nu} \right) \delta \mathcal{G}^{\mu\nu}. \quad (2.21)$$

In the Palatini formulation of gravity, the contribution to Θ coming from the bulk term is obtained from the variation of the Riemann tensor $\delta \hat{R}^\alpha_{\beta\mu\nu} = \nabla_\mu \delta \hat{\Gamma}^\alpha_{\beta\nu} - \nabla_\nu \delta \hat{\Gamma}^\alpha_{\beta\mu}$. For the radial foliation (2.2), the surface term will involve only certain components of the connection $\hat{\Gamma}^\alpha_{\mu\nu}$, related to the extrinsic curvature as

$$\hat{\Gamma}^\rho_{ij} = \frac{1}{N} K_{ij}, \quad \hat{\Gamma}^i_{\rho j} = -N K_j^i, \quad (2.22)$$

such that it can be written as

$$\Theta = -\varepsilon_{a_1 a_2 \dots a_d} \delta K^{a_1} e^{a_2} \dots e^{a_d} + c_d \delta B_d. \quad (2.23)$$

The same coordinates frame implies that the components of the Lorentz curvature \hat{R}^{AB} projected on ∂M are

$$\hat{R}^{1a} = DK^a = D_i K_j^a dx^i \wedge dx^j, \quad (2.24)$$

$$\hat{R}^{ab} = R^{ab} - K^a K^b = \left(\frac{1}{2} R_{ij}^{ab} - K_i^a K_j^b \right) dx^i \wedge dx^j \quad (2.25)$$

where D_i the covariant derivative in the boundary indices (defined with the submanifold spin connection ω^{ab}) and we have dropped the components along $d\rho$. eqs. (2.24) and (2.25) are just the well-known Gauss-Coddazzi relations for a radial foliation (2.2)

$$\hat{R}_{ij}^{\rho l} = -\frac{1}{N} \nabla_{[i} K_{j]}^l, \quad (2.26)$$

$$\hat{R}_{ij}^{kl} = R_{ij}^{kl} - K_i^k K_j^l + K_i^l K_j^k. \quad (2.27)$$

As an illustrative, simple example of the present procedure, in the next section we show explicitly how the Kounterterms series in five dimensions leads to a well-posed action principle for boundary conditions derived from the asymptotic form for AAdS spacetimes (1.1), (1.2).

3. Five-dimensional case

Dirichlet counterterms are local functional that preserve general covariance at the boundary. Kounterterms respect Lorentz-covariance in the tangent space, what provides a criterion to select them. From the surface term (2.23), it is clear that B_d must be constructed with the same parity as the bulk action, that is, the same invariant tensor $\epsilon_{A_1 \dots A_D}$ for the Lorentz group (and not $\delta_{[CD]}^{[AB]}$). In particular, this argument rules out the addition of topological invariants of the Pontryagin class in even dimensions and many possible boundary terms in the case we are treating here.

Kounterterms are built up as totally antisymmetric $2n$ -forms. This eliminates a possible inclusion of terms containing covariant derivatives of the intrinsic curvature, which are discarded by the Bianchi identity $\nabla_{[m}R_{ij]}^{kl} = 0$.

The extrinsic curvature can be defined in an arbitrary frame as $K_{AB} = -h_A^C h_B^D n_{C;D}$, where n^A is a unit vector normal to the boundary, and related to the SFF by $\theta^{AB} = n^A K^B - n^B K^A$, ($K^A = K_B^A e^B$). In that way, we can always write down the Kounterterms as a fully-covariant expression, independent of any particular foliation. In five dimensions this is given by

$$B_4 = \epsilon_{A_1 \dots A_5} \theta^{A_1 A_2} e^{A_3} \left(R^{A_4 A_5} + \frac{1}{2} \theta^{A_4}{}^C \theta^{C A_5} + \frac{1}{6\ell^2} e^{A_4} e^{A_5} \right), \quad (3.1)$$

with the equivalence in tensorial notation (see appendix B)

$$B_4 = \sqrt{-h} \delta_{[j_1 j_2 j_3]}^{[i_1 i_2 i_3]} K_{i_1}^{j_1} \left(R_{i_2 i_3}^{j_2 j_3} - K_{i_2}^{j_2} K_{i_3}^{j_3} + \frac{1}{3\ell^2} \delta_{i_2}^{j_2} \delta_{i_3}^{j_3} \right). \quad (3.2)$$

3.1 Variational principle and asymptotic conditions

Arbitrary variations of the total action (2.1) produce a surface term

$$\begin{aligned} \delta I_5 = & -2 \int_{\partial M} \kappa_5 \varepsilon_{abcd} \delta K^a e^b e^c e^d + c_4 \varepsilon_{abcd} \delta K^a e^b \left(R^{cd} - \frac{3}{2} K^c K^d + \frac{1}{6\ell^2} e^c e^d \right) \\ & + c_4 \varepsilon_{abcd} K^a \delta e^b \left(R^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right). \end{aligned} \quad (3.3)$$

when equations of motion hold. The constant in front of the bulk action is $\kappa_5 = 1/(96\pi G)$. The above equation can be conveniently rewritten as

$$\delta I_5 = -2 \int_{\partial M} \kappa_5 \varepsilon_{abcd} \delta K^a e^b e^c e^d + 2c_4 \varepsilon_{abcd} \delta K^a e^b \left(\hat{R}^{cd} + \frac{1}{3\ell^2} e^c e^d \right) \quad (3.4)$$

$$-c_4 \varepsilon_{abcd} (\delta K^a e^b - K^a \delta e^b) \left(R^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right), \quad (3.5)$$

with the help of the Gauss-Coddazzi relation (2.25).

A well-posed variational principle for precise asymptotic conditions in AdS gravity is essential to attain the finiteness of the conserved quantities and the Euclidean action. This amounts to on-shell cancelation of the surface term (3.3) using boundary conditions derived from the asymptotic form of AAdS spacetimes (1.1), (1.2).

In general, the Dirichlet variational problem for gravity is well-defined if one supplements the action by the Gibbons-Hawking term and fixes the metric h_{ij} at the boundary. In this way, the surface term coming from an arbitrary variation of the Dirichlet action vanishes identically, no matter if the boundary is at a finite distance (e.g., on a brane, to define the Israel matching conditions) or at infinity. However, it has been recently argued in ref. [11] that the standard Dirichlet boundary condition for the metric h_{ij} does not really make sense for manifolds with conformal boundary, as it is the case of AAdS spaces. It is clear from eqs. (1.1), (1.2) that the variation of h_{ij} is divergent at $\rho = 0$ and one should

instead fix the conformal structure $g_{(0)ij}$. However, due to the divergence produced in this way, the action requires additional boundary terms on top of the Gibbons-Hawking term, that turn out to be the standard Dirichlet counterterms.

In the present formulation, we will consider a boundary condition that derives from the asymptotic expansion of the extrinsic curvature

$$K_j^i = K_{jl}h^{li} = \frac{1}{\ell}\delta_j^i - \frac{\rho}{\ell}(g_{(0)}^{-1}g_{(1)})_j^i - \frac{\rho^2}{\ell}(2g_{(0)}^{-1}g_{(2)} - g_{(0)}^{-1}g_{(1)}g_{(0)}^{-1}g_{(1)})_j^i + \dots, \quad (3.6)$$

in increasing powers of ρ . This asymptotic behavior implies that the extrinsic curvature satisfies

$$K_i^j = \frac{1}{\ell}\delta_i^j, \quad (3.7)$$

on the conformal boundary, and an arbitrary variation is given by

$$\delta K_i^j = 0. \quad (3.8)$$

Had we attempted to fix $K_{ij} = \frac{1}{\ell}\frac{g_{(0)ij}}{\rho} + \dots$, we would have faced the same problem as fixing the boundary metric h_{ij} . On the contrary, eq. (3.8) is a regular boundary condition on the extrinsic curvature that can be derived from fixing $g_{(0)ij}$ due to the asymptotic form of AAdS spaces. In particular, in the asymptotically flat limit ($\ell \rightarrow \infty$) this accident no longer occurs.

The condition (3.7) has been also taken as the boundary data for the problem of holographic reconstruction of the spacetime in terms of the extrinsic curvature in ref. [11].

Making explicit the indices in the term proportional to the *curl* $\varepsilon_{abcd}(\delta K^a e^b - K^a \delta e^b)$, the last line of eq. (3.5) is

$$\varepsilon_{abcd}\varepsilon^{i_1i_2i_3i_4}\left[\delta K_{i_1}^j e_j^a e_{i_2}^b + \delta e_j^a e_l^b \left(K_{i_1}^j \delta_{i_2}^l - K_{i_2}^l \delta_{i_1}^j\right)\right]\left(R_{i_3i_4}^{cd} - K_{i_3}^c K_{i_4}^d + \frac{1}{\ell^2}e_{i_3}^c e_{i_4}^d\right)d^4x, \quad (3.9)$$

that vanishes identically when we take the condition (3.7) on the extrinsic curvature and its variation (3.8). We assume a constant (negative) curvature in the asymptotic region (1.4), that in the language of differential forms reads

$$\hat{R}^{AB} + \frac{1}{\ell^2}e^A e^B = 0, \quad (3.10)$$

that in particular holds for the boundary indices. This is a local condition at the boundary known as **ALAdS** (asymptotically locally AdS) that in principle does not impose further restrictions on the global topology of the spacetime. Therefore, solutions of this class include not only point-like black holes as Schwarzschild-AdS and Kerr-AdS, but also extended objects as black strings.

The coupling constant c_4 is then fixed as

$$c_4 = \frac{3\kappa_5\ell^2}{4} = \frac{\ell^2}{128\pi G}, \quad (3.11)$$

to cancel the rest of the surface term.

Now that we know the coefficient in front of the boundary term, we notice that the term proportional to $\sqrt{-h}K$ carries an *anomalous* factor $\frac{1}{64\pi G}$ compared to the one of the Gibbons-Hawking term in eq. (1.3), what is a consequence of a different action principle.

3.2 Conserved quantities

In the standard Dirichlet problem of gravity, the bulk contribution to (3.5) is canceled by the Gibbons-Hawking term and no other terms along δK^a can appear from the boundary term \mathcal{B}_{ct} , as it is a functional only of intrinsic quantities.

In the present approach, the surface term (3.5) contains variations along the extrinsic curvature, such that we cannot identify a quasilocal (boundary) stress tensor from the variation of the total action.

However, we can always define the energy and other conserved charges associated to asymptotic symmetries of a gravitational system through the Noether theorem.

In the appendix C we summarize the construction of the Noether charges for an arbitrary Lagrangian. The conserved current in the case we are considering here is given by

$$*J = -\Theta(e^a, K^a, \delta e^a, \delta K^a) - i_\xi(L_5 + c_4 dB_4), \quad (3.12)$$

where L_5 is the bulk Lagrangian, i_ξ is the interior derivative (also known as contraction operator) with the Killing vector ξ^μ defined in the appendix C and Θ is the surface term

$$\begin{aligned} \Theta = & \frac{\ell^2}{32\pi G} \left[\epsilon_{abcd} \delta K^a e^b \left(\hat{R}^{cd} + \frac{1}{\ell^2} e^c e^d \right) \right. \\ & \left. - \frac{1}{2} \epsilon_{abcd} (\delta K^a e^b - K^a \delta e^b) \left(R^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right) \right]. \end{aligned} \quad (3.13)$$

The derivation of the conserved charges from the current extensively use the properties of interior, exterior and Lie derivatives, Bianchi identity, and the equations of motion in differential forms language. However, we will exploit a shortcut for the charges (C.7) pointed out in appendix C, by identifying the contributions coming from the bulk Lagrangian and the boundary term B_4 .

In doing so, the Noether charge is written as

$$Q(\xi) = \mathcal{K}(\xi) + c_4 \int_{\partial\Sigma} \left(i_\xi K^a \frac{\delta B_4}{\delta K^a} + i_\xi e^a \frac{\delta B_4}{\delta e^a} \right) \quad (3.14)$$

where the first term is known as the Komar's integral

$$\mathcal{K}(\xi) = \frac{1}{48\pi G} \int_{\partial\Sigma} \epsilon_{abcd} i_\xi K^a e^b e^c e^d, \quad (3.15)$$

and it is the conserved quantity associated to the bulk term in the gravity action.²

²In an arbitrary dimension $D > 3$, the Komar's integral has the form

$$\mathcal{K}(\xi) = \frac{1}{8\pi G_N} \int_{\partial\Sigma} \epsilon_{a_1 \dots a_{D-1}} i_\xi K^{a_1} e^{a_2} \dots e^{a_{D-1}} = \frac{1}{16\pi G_N} \int_{\partial\Sigma} \nabla^\mu \xi^\nu d\Sigma_{\mu\nu},$$

where $d\Sigma_{\mu\nu} = \frac{\sqrt{\sigma}}{(D-2)!} \epsilon_{\mu\nu\alpha_1 \dots \alpha_{D-2}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{D-2}}$ is the dual of the area $(D-2)$ -form (σ is the determinant of the metric on the sphere S^{D-2}). When evaluated for a timelike Killing vector on the Schwarzschild-AdS solution, this formula gives a factor $(D-3)/(D-2)$ times the mass, plus a divergence in the radial coordinate. The divergence is usually canceled by background subtraction, procedure that however does not solve the problem of the anomalous Komar factor [26].

The expression for the Noether charge appears to be naturally split in two parts, associated to the first and second line of the surface term (3.13), respectively

$$Q(\xi) = q(\xi) + q_0(\xi), \tag{3.16}$$

$$q(\xi) = \frac{\ell^2}{32\pi G} \int_{\partial\Sigma} \epsilon_{abcd} i_\xi K^a e^b \left(\hat{R}^{cd} + \frac{1}{\ell^2} e^c e^d \right), \tag{3.17}$$

$$q_0(\xi) = -\frac{\ell^2}{64\pi G} \int_{\partial\Sigma} \epsilon_{abcd} \left(i_\xi K^a e^b + K^a i_\xi e^b \right) \left(R^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right). \tag{3.18}$$

As we shall see below for concrete examples, the formula for $q(\xi)$ provides the standard conserved quantities (mass, angular momentum) for AAdS solutions. In tensorial notation, $q(\xi)$ takes the form

$$q(\xi) = \frac{\ell^2}{64\pi G} \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 \dots i_4} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left(\hat{R}_{j_3 j_4}^{i_3 i_4} + \frac{1}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) dx^{j_2} dx^{j_3} dx^{j_4}, \tag{3.19}$$

where the product $dx^{j_2} \wedge dx^{j_3} \wedge dx^{j_4}$ stands for the volume element of the boundary of the spatial section $\partial\Sigma$ (i.e., boundary indices without the time). Notice that (3.19) is proportional to the l.h.s. of eq. (1.4), and therefore vanishes identically for any solution of global constant curvature, as AdS vacuum and spacetimes with topological identifications that preserve AdS flatness.

The above reasoning leads us to consider (3.18) as a general formula for the vacuum energy in five-dimensional AAdS spacetimes

$$q_0(\xi) = -\frac{\ell^2}{128\pi G} \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 i_2 i_3 i_4} \xi^k \left(K_k^{i_1} \delta_{j_2}^{i_2} + K_{j_2}^{i_1} \delta_k^{i_2} \right) \left(R_{j_3 j_4}^{i_3 i_4} - \frac{1}{2} K_{[j_3 j_4]}^{[i_3 i_4]} + \frac{1}{2\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) dx^{j_2} dx^{j_3} dx^{j_4}, \tag{3.20}$$

where we have introduced the shorthand

$$K_{[kl]}^{[ij]} = K_k^i K_l^j - K_k^j K_l^i. \tag{3.21}$$

The regularization of the conserved quantities and the Euclidean action in five-dimensional AAdS solutions is illustrated through concrete examples below.

3.3 Examples

Schwarzschild-AdS black hole and topological extensions. Static solutions to EH-AdS gravity are given by Schwarzschild black hole metric. Due to the negative cosmological constant, the transversal section Σ_{D-2}^k can be the sphere S^{D-2} , a $(D-2)$ -dim locally flat space or the hyperboloid H^{D-2}

$$ds^2 = -\Delta^2(r) dt^2 + \frac{dr^2}{\Delta^2(r)} + r^2 \gamma_{mn} d\theta^m d\theta^n, \tag{3.22}$$

where the metric function is

$$\Delta^2(r) = k - \frac{2\omega_D G \mu}{r^{D-3}} + \frac{r^2}{\ell^2}. \tag{3.23}$$

Here, μ appears as an integration constant, $\omega_D = \frac{8\pi}{(D-2)\text{Vol}(S^{D-2})}$ and $\gamma_{\underline{m}\underline{n}}$ ($\underline{m}, \underline{n} = 1, \dots, D-2$) is the metric of Σ_{D-2}^k of constant curvature $k = \pm 1, 0$. The event horizon r_+ is defined as the largest root of $\Delta^2(r_+) = 0$.

In order to compute the mass and the vacuum energy, we use the following components of the extrinsic and intrinsic curvatures

$$K_t^t = -\Delta', \quad K_{\underline{m}}^{\underline{n}} = -\frac{\Delta}{r}\delta_{\underline{m}}^{\underline{n}}, \quad (3.24)$$

$$R_{\underline{m}_2\underline{n}_2}^{\underline{m}_1\underline{n}_1} = \frac{k}{r^2}\delta_{[\underline{m}_2\underline{n}_2]}^{[\underline{m}_1\underline{n}_1]}. \quad (3.25)$$

where prime denotes the derivative d/dr , and the determinant of the boundary metric

$$\sqrt{-h} = \Delta\sqrt{\gamma}r^{D-2}. \quad (3.26)$$

Notice the opposite sign in the leading order of the asymptotic behavior of K_j^i respect to eq. (3.6), due to the change in the radial coordinate.

Thus, for the time-like Killing vector $\xi = \partial/\partial t$, formula 3.19) gives

$$q(\partial_t) = M = 3!\text{Vol}(\Sigma_3^k) \lim_{r \rightarrow \infty} (\Delta^2)'r \left\{ \kappa_3 r^2 + 2c_4 \left(k - \Delta^2 + \frac{r^2}{3\ell^2} \right) \right\}, \quad (3.27)$$

$$= \frac{\text{Vol}(\Sigma_3^k)}{\text{Vol}(S^3)}\mu, \quad (3.28)$$

in agreement with the result from background-dependent methods, e.g., Hamiltonian formalism [27]. We have written eq. (3.27) as an intermediate step just to make clear the contributions from the bulk and the boundary that will be useful for the black hole thermodynamics below.

Plugging the metric (3.22) into the formula (3.20), we obtain

$$q_0(\partial_t) = E_0 = -3!\text{Vol}(\Sigma_3^k)c_4 \lim_{r \rightarrow \infty} \left(\Delta^2 - \frac{r(\Delta^2)'}{2} \right) \left(k - \frac{1}{2}\Delta^2 + \frac{r^2}{2\ell^2} \right) \quad (3.29)$$

$$= (-k)^2 \frac{3\text{Vol}(\Sigma_3^k)}{64\pi G} \ell^2. \quad (3.30)$$

The result for the spherical case ($k = 1$) is the Balasubramanian-Kraus vacuum energy $E_0 = \frac{3\pi\ell^2}{32G}$ for SAdS black hole that appears in the Dirichlet regularization of the stress tensor [7].

In the case $k = -1$, cosmic censorship holds for a mass over a critical value $M_c = \frac{\text{Vol}(H^3)}{\text{Vol}(S^3)}\mu_c$ that separates black holes with hyperbolic transversal section from naked singularities. This value, for an arbitrary dimension, is

$$\mu_c = -\frac{\ell^{D-3}}{\omega_D G} \sqrt{\frac{(D-3)^{D-3}}{(D-1)^{D-1}}}, \quad (3.31)$$

where $\omega_D = \frac{8\pi}{(D-2)\text{Vol}(S^{D-2})}$. The critical mass in five dimensions has the same value (with opposite sign) as the vacuum energy (3.30). Therefore, despite the fact that in 5D AdS

gravity there exist black holes with negative mass, the vacuum energy restores the positivity of the total energy $E = M + E_0$ [8].

We show now that the boundary term (3.2) makes finite the Euclidean action (2.1) for five-dimensional SAdS solution, and that correctly describes black hole thermodynamics.

The Euclidean period β is defined as

$$\beta = T^{-1} = \frac{4\pi}{(\Delta^2)' \Big|_{r_+}} \tag{3.32}$$

where T is the black hole temperature. This condition comes from the requirement that, in the Euclidean sector the solution (3.22) does not have a conical singularity at the coordinates origin ($r = r_+$). In the canonical ensemble, the Euclidean action

$$S = \beta\mathcal{E} - I^E, \tag{3.33}$$

defines the entropy S and the thermodynamic energy

$$\mathcal{E} = -\frac{\partial I^E}{\partial \beta} \tag{3.34}$$

of a black hole for a fixed temperature. The Euclidean bulk action is evaluated for a static black hole of the form (3.22) as a total derivative in the radial coordinate, such that

$$I_{\text{bulk}}^E = -\kappa_5 3! \text{Vol}(\Sigma_3^k) \beta [(\Delta^2)' r^3] \Big|_{r_+}^{\infty}, \tag{3.35}$$

and the Euclidean boundary term as

$$\int_{\partial M} B_4^E = -3! \text{Vol}(\Sigma_3^k) \beta \left[r(\Delta^2)' \left(k - \Delta^2 + \frac{r^2}{3\ell^2} \right) + \left(\Delta^2 - \frac{r(\Delta^2)'}{2} \right) \left(k - \frac{1}{2}\Delta^2 + \frac{r^2}{2\ell^2} \right) \right] \Big|_{r_+}^{\infty}. \tag{3.36}$$

The total Euclidean action $I_5^E = I_{\text{bulk}}^E + c_4 \int_{\partial M} B_4^E$ contains two contributions: the first one from the bulk at radial infinity plus the boundary term that can be identified (using eqs. (3.27) and (3.29) as $-\beta(M + E_0)$.

Therefore, the finiteness of the Noether charges for static black holes ensures that the divergencies at $r = \infty$ of the bulk Euclidean action are exactly canceled by the ones in the boundary term B_4 .

It is reassuring to check that the thermodynamic energy definition

$$\mathcal{E} = -\frac{\partial I^E / \partial r_+}{\partial \beta / \partial r_+} = M + E_0, \tag{3.37}$$

recovers the same result for the total energy as from the Noether charges defined above.

Finally, the entropy is the horizon contribution of the bulk Euclidean action

$$S = \frac{\text{vol}(\Sigma_3^k) r_+^3}{4G} = \frac{\text{Area}}{4G}. \tag{3.38}$$

-Kerr-AdS black hole. The general Kerr-AdS metric in five dimensions possesses two rotation parameters a, b , and it can be written in Boyer-Lindquist coordinates as [28]

$$\begin{aligned}
 ds^2 = & -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \\
 & + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{(r^2 + a^2)}{\Xi_a} d\phi \right)^2 + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(b dt - \frac{(r^2 + b^2)}{\Xi_b} d\psi \right)^2 + \\
 & + \frac{(1 + r^2/\ell^2)}{r^2 \rho^2} \left(abdt - \frac{b(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi - \frac{a(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\psi \right)^2, \quad (3.39)
 \end{aligned}$$

where the functions in the metric are

$$\Delta_r \equiv \frac{1}{r^2} (r^2 + a^2) (r^2 + b^2) (1 + r^2/\ell^2) - 2m, \quad (3.40)$$

$$\Delta_\theta \equiv 1 - \frac{a^2}{\ell^2} \cos^2 \theta - \frac{b^2}{\ell^2} \sin^2 \theta, \quad (3.41)$$

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad (3.42)$$

$$\Xi_a \equiv 1 - \frac{a^2}{\ell^2}, \quad \Xi_b \equiv 1 - \frac{b^2}{\ell^2}, \quad (3.43)$$

$$0 \leq \theta \leq \pi/2, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi. \quad (3.44)$$

The event horizon r_+ is the largest solution of the equation $\Delta_r(r_+) = 0$, whose area is

$$Area = \frac{2\pi^2 (r_+^2 + a^2) (r_+^2 + b^2)}{r_+ \Xi_a \Xi_b}. \quad (3.45)$$

Evaluating the charge formula (3.19) for the metric (3.39)–(3.44), we have

$$E' = q(\partial_t) = \frac{3\pi}{4G} \frac{m}{\Xi_a \Xi_b}, \quad (3.46)$$

$$E_0 = q_0(\partial_t) = \frac{\pi \ell^2}{96 \Xi_a \Xi_b G} (7 \Xi_a \Xi_b + \Xi_a^2 + \Xi_b^2), \quad (3.47)$$

and the angular momenta

$$J_a = q(\partial_\phi) = \frac{\pi}{2G} \frac{ma}{\Xi_a^2 \Xi_b}, \quad (3.48)$$

$$J_b = q(\partial_\psi) = \frac{\pi}{2G} \frac{mb}{\Xi_a \Xi_b^2}. \quad (3.49)$$

The quantity (3.46) is the energy obtained by Awad and Johnson in [29] that, however, does not satisfy the first law of black hole thermodynamics, as has been pointed out in [30]. The expression for the vacuum energy E_0 is in agreement with [29], and it is equivalent to the Papadimitriou-Skenderis result [11]

$$E_0 = \frac{3\pi \ell^2}{32G} \left(1 + \frac{(\Xi_a - \Xi_b)^2}{9 \Xi_a \Xi_b} \right), \quad (3.50)$$

both computed using different versions of the counterterms method.

Notice that the physical energy for Kerr-AdS is obtained with a Killing vector that does not rotate at infinity $\xi = \partial_t + \frac{a}{\ell^2} \partial_\phi + \frac{b}{\ell^2} \partial_\psi$,

$$E = q \left(\partial_t + \frac{a}{\ell^2} \partial_\phi + \frac{b}{\ell^2} \partial_\psi \right) = \frac{\pi m}{4G \Xi_a^2 \Xi_b^2} (2\Xi_a + 2\Xi_b - \Xi_a \Xi_b). \quad (3.51)$$

The on-shell Euclidean action is

$$I_5^E = \frac{\beta}{2\pi G \ell^2} \int_{r_+}^{\infty} dr \int d\Omega N \sqrt{-h} + \beta c_4 \int d\Omega (B_4)|_{r=\infty} \quad (3.52)$$

and we see that the boundary term cancels out the divergences coming from the bulk action, such that we have the finite result

$$I_5^E = \frac{\beta\pi}{96\ell^2 \Xi_a \Xi_b G} (-a^4 + 9a^2\ell^2 + 24r_+^2 a^2 + 17a^2 b^2 - 24m\ell^2 - 9\ell^4 + 24r_+^4 - b^4 + 9\ell^2 b^2 + 24r_+^2 b^2), \quad (3.53)$$

This expression can also be put into the form of Awad-Johnson

$$I_5^E = \frac{\beta\pi}{96\ell^2 \Xi_a \Xi_b G} (12 (r_+^2/\ell^2) (1 - \Xi_a - \Xi_b) + \Xi_a^2 + \Xi_b^2 + \Xi_a \Xi_b + 12r_+^4/\ell^4 - 2(a^4 + b^4)/\ell^4 - 12 (a^2 b^2/\ell^4) (\ell^2/r_+^2 - 1/3) - 12), \quad (3.54)$$

or the more compact one obtained by Papadimitriou-Skenderis

$$I_5^E = \beta E_0 + \frac{\beta\pi}{4G\ell^2 \Xi_a \Xi_b} (m\ell^2 - (r_+^2 + a^2)(r_+^2 + b^2)). \quad (3.55)$$

In order to obtain the correct value for the entropy, one must use $\mathcal{E} = E + E_0$ as the total energy for the black hole system, the angular velocities respect to a non-rotating frame at infinity

$$\Omega_a = \frac{a(1 + r_+^2 \ell^{-2})}{r_+^2 + a^2}, \quad \Omega_b = \frac{b(1 + r_+^2 \ell^{-2})}{r_+^2 + b^2}, \quad (3.56)$$

and the Euclidean period

$$\beta = \frac{2\pi(r_+^2 + a^2)(r_+^2 + b^2)\ell^2}{2r_+^6 + r_+^4(\ell^2 + b^2 + a^2) - a^2 b^2 \ell^2}. \quad (3.57)$$

The above quantities satisfy the thermodynamical relation

$$S = \beta(\mathcal{E} - \Omega_a J_a - \Omega_b J_b) - I_5^E = \frac{Area}{4G} \quad (3.58)$$

that recovers the entropy in terms of the area for Kerr-AdS black hole (3.45).

-Clarkson-Mann solitons. In recent papers [31, 32], new solitons in cosmological spacetimes were presented. These solutions resemble the Eguchi-Hanson metrics in four dimensions [33] and in the case of a negative cosmological constant, they possess AdS/Zp asymptotics and a lower energy than the global AdS or even global AdS/Zp spacetimes.

The Clarkson-Mann-AdS soliton metric reads

$$ds^2 = -g(r)dt^2 + \frac{r^2 f(r)}{4} [d\psi + \cos\theta d\phi]^2 + \frac{dr^2}{f(r)g(r)} + \frac{r^2}{4} d\Omega_2^2,$$

$$g(r) = 1 + \frac{r^2}{\ell^2}, \quad f(r) = 1 - \frac{a^4}{r^4}, \tag{3.59}$$

where $d\Omega_2^2$ is the metric of the unit 2-sphere.

In order to remove the stringlike singularity at $r = a$, the period of ψ must be $4\pi/p$ and the parameter a satisfies the relation

$$a^2 = \ell^2 \left(\frac{p^2}{4} - 1 \right), \tag{3.60}$$

with $p \geq 3$.

The energy for this solitonic solution is negative

$$E = q(\partial_t) = -\frac{\pi a^4}{8G\ell^2 p} \tag{3.61}$$

in agreement to the result computed using the standard counterterms procedure in [31, 34]. The present method also reproduces the value of the vacuum energy, which is lower than that of global AdS spacetime

$$E_0 = q_0(\partial_t) = \frac{3\pi\ell^2}{32Gp}. \tag{3.62}$$

The negative mass (3.61) has also been found in [35] through a spin-connection formulation of the Abbott-Deser [36] (and, more recently, Deser-Tekin [37, 38]) method.

The total Euclidean action

$$I^E = I_{\text{bulk}}^E + c_4 \int_{\partial M} B_4^E, \tag{3.63}$$

turns out to be

$$I^E = \beta(E + E_0) \tag{3.64}$$

where the Euclidean period β remains arbitrary, as the solution is horizonless. As a consequence, the entropy of the system is zero.

4. Seven-dimensional case

Before we go into the general odd-dimensional case, let us consider $D = 7$ also for illustrative purposes. The expression for the boundary term

$$B_6 = -6 \int_0^1 dt \int_0^t ds \varepsilon_{a_1 \dots a_6} K^{a_1} e^{a_2} \left(R^{a_3 a_4} - t^2 K^{a_3} K^{a_4} + \frac{s^2}{\ell^2} e^{a_3} e^{a_4} \right) \times$$

$$\times \left(R^{a_5 a_6} - t^2 K^{a_5} K^{a_6} + \frac{s^2}{\ell^2} e^{a_5} e^{a_6} \right), \tag{4.1}$$

after the parametric integrations are performed is given by

$$\begin{aligned}
 B_6 = & -3\varepsilon_{a_1\dots a_6} K^{a_1} e^{a_2} \left(R^{a_3 a_4} R^{a_5 a_6} - \frac{1}{3} K^{a_3} K^{a_4} K^{a_5} K^{a_6} + \frac{1}{15\ell^4} e^{a_3} e^{a_4} e^{a_5} e^{a_6} \right. \\
 & \left. - R^{a_3 a_4} K^{a_5} K^{a_6} + \frac{1}{3\ell^2} R^{a_3 a_4} e^{a_5} e^{a_6} - \frac{2}{9\ell^2} K^{a_3} K^{a_4} e^{a_5} e^{a_6} \right). \quad (4.2)
 \end{aligned}$$

The equivalent tensorial form of the Kounterterms for seven dimensions is

$$\begin{aligned}
 B_6 = & \frac{3}{4} \sqrt{-h} \delta_{[j_1\dots j_5]}^{[i_1\dots i_5]} K_{i_1}^{j_1} \left(R_{i_2 i_3}^{j_2 j_3} R_{i_4 i_5}^{j_4 j_5} - \frac{4}{3} K_{i_2}^{j_2} K_{i_3}^{j_3} K_{i_4}^{j_4} K_{i_5}^{j_5} + \frac{4}{15\ell^4} \delta_{i_2}^{j_2} \delta_{i_3}^{j_3} \delta_{i_4}^{j_4} \delta_{i_5}^{j_5} \right. \\
 & \left. - 2R_{i_2 i_3}^{j_2 j_3} K_{i_4}^{j_4} K_{i_5}^{j_5} + \frac{2}{3\ell^2} R_{i_2 i_3}^{j_2 j_3} \delta_{i_4}^{j_4} \delta_{i_5}^{j_5} - \frac{8}{9\ell^2} K_{i_2}^{j_2} K_{i_3}^{j_3} \delta_{i_4}^{j_4} \delta_{i_5}^{j_5} \right). \quad (4.3)
 \end{aligned}$$

4.1 Action principle

Now we develop a similar treatment as in the five-dimensional case, writing down an adequate form of the variation of the boundary term, in order to use suitable boundary conditions for AAdS spacetimes. Varying the seven-dimensional action, we have — on-shell — a total surface term

$$\begin{aligned}
 \delta I_7 = & -2 \int_{\partial M} \kappa_7 \epsilon_{abcdefg} \delta K^a e^b e^c e^d e^f e^g + \frac{3}{2} c_6 \epsilon_{abcdefg} \delta K^a e^b \left(R^{cd} R^{fg} + \frac{5}{3} K^c K^d K^f K^g \right. \\
 & \left. + \frac{1}{15\ell^4} e^c e^d e^f e^g - 3R^{cd} K^f K^g + \frac{1}{3\ell^2} R^{cd} e^f e^g - \frac{2}{3\ell^2} K^c K^d e^f e^g \right) + \frac{3}{2} c_6 \epsilon_{abcdefg} K^a \delta e^b \times \\
 & \times \left(R^{cd} R^{fg} + \frac{1}{3} K^c K^d K^f K^g + \frac{1}{3\ell^4} e^c e^d e^f e^g - R^{cd} K^f K^g + \frac{1}{\ell^2} R^{cd} e^f e^g - \frac{2}{3\ell^2} K^c K^d e^f e^g \right). \quad (4.4)
 \end{aligned}$$

The constant in front of the bulk action is $\kappa_7 = 1/(16\pi G \times 5!)$. The total variation can be put into the form

$$\begin{aligned}
 \delta I_7 = & -2 \int_{\partial M} \kappa_7 \epsilon_{abcdefg} \delta K^a e^b e^c e^d e^f e^g + 3c_6 \epsilon_{abcdefg} \delta K^a e^b \left(\hat{R}^{cd} \hat{R}^{fg} + \frac{2}{3\ell^2} \hat{R}^{cd} e^f e^g + \frac{1}{5\ell^4} e^c e^d e^f e^g \right) \\
 & - \frac{3}{2} c_6 \epsilon_{abcdefg} (\delta K^a e^b - K^a \delta e^b) \left(R^{cd} R^{fg} + \frac{1}{3} K^c K^d K^f K^g + \frac{1}{3\ell^4} e^c e^d e^f e^g - R^{cd} K^f K^g \right. \\
 & \left. + \frac{1}{\ell^2} R^{cd} e^f e^g - \frac{2}{3\ell^2} K^c K^d e^f e^g \right). \quad (4.5)
 \end{aligned}$$

using again the Gauss-Coddazzi relation (2.25). The above relation already hints a pattern for the surface term coming from the total variation of the action,

$$\begin{aligned}
 \delta I_7 = & -2 \int_{\partial M} \kappa_7 \epsilon_{abcdefg} \delta K^a e^b e^c e^d e^f e^g + 3c_6 \int_0^1 dt \epsilon_{abcdefg} \delta K^a e^b \left(\hat{R}^{cd} + \frac{t^2}{\ell^2} e^c e^d \right) \left(\hat{R}^{fg} + \frac{t^2}{\ell^2} e^f e^g \right) \\
 & - 3c_6 \int_0^1 dt t \epsilon_{abcdefg} (\delta K^a e^b - K^a \delta e^b) \left(R^{cd} - t^2 K^c K^d + \frac{t^2}{\ell^2} e^c e^d \right) \left(R^{fg} - t^2 K^f K^g + \frac{t^2}{\ell^2} e^f e^g \right) \quad (4.6)
 \end{aligned}$$

that we will confirm below in the general odd-dimensional case.

As in the five-dimensional case, we consider a more explicit form of the second line

$$\begin{aligned} & \varepsilon_{abcdefg} \varepsilon^{i_1 i_2 i_3 i_4 i_5 i_6} \left[\delta K_{i_1}^j e_j^a e_{i_2}^b + \delta e_j^a e_l^b \left(K_{i_1}^j \delta_{i_2}^l - K_{i_2}^l \delta_{i_1}^j \right) \right] \times \\ & \times \int_0^1 dt t \left(\frac{1}{2} R_{i_3 i_4}^{cd} - t^2 K_{i_3}^c K_{i_4}^d + \frac{t^2}{\ell^2} e_{i_3}^c e_{i_4}^d \right) \left(\frac{1}{2} R_{i_5 i_6}^{fg} - t^2 K_{i_5}^f K_{i_6}^g + \frac{t^2}{\ell^2} e_{i_5}^f e_{i_6}^g \right) d^6 x, \end{aligned} \quad (4.7)$$

that again vanishes identically for the boundary condition on the extrinsic curvature (3.7) and the corresponding variation (3.8). Thus, the problem of a well-defined action principle amounts to fixing the coupling of the boundary term c_6 . In the asymptotic region the spacetime curvature is constant and then, inserting (3.10) in the first line of eq. (4.6), we have

$$\delta I_7 = -2 \int_{\partial M} \varepsilon_{abcdefg} \delta K^a e^b e^c e^d e^f e^g \left(\kappa_7 + 3 \frac{c_6}{\ell^4} \int_0^1 dt (-1 + t^2)^2 \right). \quad (4.8)$$

Thus, the cancelation of the surface term implies

$$c_6 = -\frac{5}{8} \kappa_7 \ell^4 = -\frac{\ell^4}{16\pi G \times 192}. \quad (4.9)$$

Now that we have achieved a well-posed action principle, we discuss the construction of the Noether charges that derive from it.

4.2 Conserved charges

The Noether current for the seven-dimensional case is

$$*J = -\Theta(e^a, K^a, \delta e^a, \delta K^a) - i_\xi (L_7 + c_6 dB_6), \quad (4.10)$$

where L_7 is the bulk Lagrangian and Θ is the surface term

$$\begin{aligned} \Theta = & -2\kappa_7 \int_{\partial M} \varepsilon_{abcdefg} \delta K^a e^b \left[e^c e^d e^f e^g - \frac{15}{8} \ell^4 \int_0^1 dt \left(\hat{R}^{cd} + \frac{t^2}{\ell^2} e^c e^d \right) \left(\hat{R}^{fg} + \frac{t^2}{\ell^2} e^f e^g \right) \right] \\ & + \frac{15}{8} \ell^4 \int_0^1 dt t \varepsilon_{abcdefg} (\delta K^a e^b - K^a \delta e^b) \left(R^{cd} - t^2 K^c K^d + \frac{t^2}{\ell^2} e^c e^d \right) \left(R^{fg} - t^2 K^f K^g + \frac{t^2}{\ell^2} e^f e^g \right) \end{aligned} \quad (4.11)$$

Carrying out the construction in appendix C for a bulk Lagrangian supplemented in a boundary term, the Noether charge is written as

$$Q(\xi) = \int_{\partial \Sigma} 2\kappa_7 \varepsilon_{abcdefg} i_\xi K^a e^b e^c e^d e^f e^g + c_6 \left(i_\xi K^a \frac{\delta B_6}{\delta K^a} + i_\xi e^a \frac{\delta B_6}{\delta e^a} \right) \quad (4.12)$$

The formula for the conserved quantities contains two contributions

$$Q(\xi) = q(\xi) + q_0(\xi), \quad (4.13)$$

that can be traced back to the first and second lines in the surface term (

$$q(\xi) = 2\kappa_7 \int_{\partial \Sigma} \varepsilon_{abcdefg} i_\xi K^a e^b \left[e^c e^d e^f e^g - \frac{15}{8} \ell^4 \int_0^1 dt \left(\hat{R}^{cd} + \frac{t^2}{\ell^2} e^c e^d \right) \left(\hat{R}^{fg} + \frac{t^2}{\ell^2} e^f e^g \right) \right], \quad (4.14)$$

will provide the mass and angular momentum for AAdS solutions. Equivalently, eq. (4.14) can be factorized as

$$q(\xi) = -\frac{30}{8}\kappa_7\ell^4 \int_{\partial\Sigma} \varepsilon_{abcdefg} i_\xi K^a e^b \left(\hat{R}^{cd} + \frac{1}{\ell^2} e^c e^d \right) \left(\hat{R}^{fg} - \frac{1}{3\ell^2} e^f e^g \right) \quad (4.15)$$

that in its tensorial form

$$q(\xi) = -\frac{\ell^4}{16\pi G \times 128} \int_{\partial\Sigma} \sqrt{-h} \varepsilon_{i_1 \dots i_6} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left(\hat{R}_{j_3 j_4}^{i_3 i_4} + \frac{1}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \left(\hat{R}_{j_5 j_6}^{i_5 i_6} - \frac{1}{3\ell^2} \delta_{[j_5 j_6]}^{[i_5 i_6]} \right) dx^{j_2} \dots dx^{j_6}, \quad (4.16)$$

is proportional to the curvature for the AdS group and, therefore, identically vanishing for (global) AdS spacetime. The product $dx^{j_2} \wedge \dots \wedge dx^{j_6}$ is the volume element of the boundary of the spatial section $\partial\Sigma$ (at constant time).

As a consequence, the additional term q_0 in the conserved quantities is responsible for the existence of a vacuum energy

$$\begin{aligned} q_0(\xi) &= \frac{30}{8}\kappa_7\ell^4 \int_{\partial\Sigma} \int_0^1 dt t \varepsilon_{abcdefg} \left(i_\xi K^a e^b + K^a i_\xi e^b \right) \left(R^{cd} - t^2 K^c K^d + \frac{t^2}{\ell^2} e^c e^d \right) \times \\ &\quad \times \left(R^{fg} - t^2 K^f K^g + \frac{t^2}{\ell^2} e^f e^g \right), \\ &= \frac{\ell^4}{16\pi G \times 128} \int_{\partial\Sigma} \int_0^1 dt t \sqrt{-h} \varepsilon_{i_1 \dots i_6} \xi^k (K_k^{i_1} \delta_{j_2}^{i_2} + K_{j_2}^{i_1} \delta_k^{i_2}) \left(R_{j_3 j_4}^{i_3 i_4} - t^2 K_{[j_3 j_4]}^{[i_3 i_4]} + \frac{t^2}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \times \\ &\quad \times \left(R_{j_5 j_6}^{i_5 i_6} - t^2 K_{[j_5 j_6]}^{[i_5 i_6]} + \frac{t^2}{\ell^2} \delta_{[j_5 j_6]}^{[i_5 i_6]} \right) dx^{j_2} \dots dx^{j_6} \end{aligned} \quad (4.17)$$

where we have kept the parametric integral because, otherwise, the expression is more involved.

4.3 Examples

Topological static black holes. For the seven-dimensional static metric specified by eqs. (3.22)-(Killing vector $\xi = \partial_t$). Using the relations (3.24)–(3.26), we obtain

$$q(\partial_t) = M = 5! \text{Vol}(\Sigma_5^k) \lim_{r \rightarrow \infty} (\Delta^2)' r \left\{ \kappa_7 \left[r^4 + 3c_6 \int_0^1 dt \left(k - \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2 \right] \right\} \quad (4.18)$$

$$= \frac{\text{Vol}(\Sigma_5^k)}{\text{Vol}(S^5)} \mu. \quad (4.19)$$

for the mass, whereas for the vacuum energy takes the negative value

$$q_0(\partial_t) = E_0 = -2 \times 5! c_6 \text{Vol}(\Sigma_5^k) \lim_{r \rightarrow \infty} \left(\Delta^2 - \frac{r(\Delta^2)'}{2} \right) \int_0^1 dt t \left(k - t^2 \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2 \quad (4.20)$$

$$= (-k)^3 \frac{5\ell^4}{128\pi G} \text{Vol}(\Sigma_5^k). \quad (4.21)$$

We have included an intermediate step in the computation of both the mass and vacuum energy, because these expressions will appear again in the evaluation of the Euclidean action.

With the Euclidean period β defined as in eq. (3.32), the Euclidean bulk action, evaluated for a static black hole of the form (3.22) in seven dimensions is

$$I_{\text{bulk}}^E = -5! \kappa_7 \text{Vol}(\Sigma_5^k) \beta [(\Delta^2)' r^5]_{r_+}^\infty. \quad (4.22)$$

The boundary is defined only at the asymptotic region and then, the Euclidean boundary term is

$$\begin{aligned} \int_{\partial M} B_6^E = & 2 \times 5! \text{Vol}(\Sigma_5^k) \beta \left[\frac{r(\Delta^2)'}{2} \int_0^1 dt \left(k - \Delta^2 + \frac{t^2 r^2}{\ell^2} \right)^2 + \right. \\ & \left. + \left(\Delta^2 - \frac{r(\Delta^2)'}{2} \right) \int_0^1 dt t \left(k - t^2 \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2 \right] \Big|_{r_+}^\infty. \end{aligned} \quad (4.23)$$

such that in the total Euclidean action

$$I_7^E = I_{\text{bulk}}^E + c_6 \int_{\partial M} B_6^E \quad (4.24)$$

the contribution at $r = \infty$ can be read off from eqs. (4.18), (

$$I_7^E = \frac{\text{Vol}(\Sigma_5^k)}{16\pi G} \beta r_+^5 (\Delta^2)'_{r_+} - \beta(M + E_0). \quad (4.25)$$

Using the definition of thermodynamic energy \mathcal{E} in eq. (3.37) –that is equivalent to the result obtained from the Noether theorem– we obtain the entropy as the Euclidean bulk action evaluated at $r = r_+$

$$S = \frac{\text{Vol}(\Sigma_5^k) r_+^5}{4G} = \frac{\text{Area}}{4G}. \quad (4.26)$$

Kerr-AdS₇ black hole. The number of independent rotation parameters for D -dimensional Kerr-AdS metric is equal to the number of Casimir invariants for the rotation group $\text{SO}(D - 1)$, which is the integer part of $(D - 1)/2$. The general rotating black hole in seven-dimensional AdS gravity then possesses three rotation parameters, but we shall consider below the particular case of a single rotation parameter to show the finiteness of the conserved quantities and Euclidean action.

The line element for the one-parameter Kerr-AdS₇ spacetime is

$$\begin{aligned} ds^2 = & -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + r^2 \cos^2 \theta d\psi^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \\ & + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{(r^2 + a^2)}{\Xi} d\phi \right)^2 + r^2 \cos^2 \theta d\Omega_3^2, \end{aligned} \quad (4.27)$$

where the functions in the metric are

$$\Delta_r = (r^2 + a^2) (1 + r^2/\ell^2) - 2m/r^2, \quad (4.28)$$

$$\Delta_\theta \equiv 1 - \frac{a^2}{\ell^2} \cos^2 \theta, \quad (4.29)$$

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Xi \equiv 1 - \frac{a^2}{\ell^2}, \quad (4.30)$$

and $d\Omega_3^2$ is the metric of the 3–sphere

$$d\Omega_3^2 = d\psi^2 + \sin^2 \psi d\eta^2 + \cos^2 \psi d\beta^2 \quad (4.31)$$

and the angles range is $\theta, \psi \in [0, \pi/2]$ and $\phi, \eta, \beta \in [0, 2\pi]$.

The area of the event horizon is

$$Area = \pi^3 \frac{r_+^3 (r_+^2 + a^2)}{\Xi}. \quad (4.32)$$

Using the charge formulas (4.16) and (4.17) for the metric (4.27)–(4.31) yields

$$E' = q(\partial_t) = \frac{5\pi^2 m}{8G \Xi}, \quad (4.33)$$

$$E_0 = q_0(\partial_t) = -\frac{\pi^2}{1280\Xi G \ell^2} (50\ell^6 - 50a^2\ell^4 + 5a^4\ell^2 + a^6), \quad (4.34)$$

and the angular momentum

$$J = q(\partial_\phi) = \frac{\pi^2 ma}{4G \Xi^2}. \quad (4.35)$$

The values in eqs. (4.33), (4.34) and (4.35) agree with the ones computed using a Dirichlet counterterms regularization [10, 29]. The vacuum energy can be also written as

$$E_0 = -\frac{5\pi^2 \ell^4}{128G} \left(1 + \frac{(1 - \Xi)^2 (6 - \Xi)}{50\Xi} \right) \quad (4.36)$$

in order to make more manifest the matching with the vacuum energy for Schwarzschild-AdS (4.21) in the non-rotating limit³ and to try to infer the general vacuum energy in the case of three different rotation parameters.

As pointed out in the five-dimensional section above, the physical energy for a Kerr-AdS black hole is the conserved quantity associated to a non-rotating asymptotic timelike Killing vector $\xi = \partial_t + \frac{a}{\ell^2} \partial_\phi$ [30]

$$E = q\left(\partial_t + \frac{a}{\ell^2} \partial_\phi\right) = \frac{m\pi^2}{8G\Xi^2} (2 + 3\Xi), \quad (4.37)$$

in agreement with the formulas in refs. [30, 39–41], specialized to a single nonvanishing rotation parameter.

In order to complete the discussion about regularization of this seven-dimensional solution, we compute the on-shell Euclidean action, that for an stationary spacetime is given by

$$I_7^E = \frac{3\beta}{4\pi G \ell^2} \int_{r_+}^{\infty} dr \int d\Omega N \sqrt{h} + \beta c_6 \int d\Omega (B_6)|^{r=\infty}, \quad (4.38)$$

where the Euclidean period is

$$\beta = \frac{2\pi(r_+^2 + a^2)r_+}{3r_+^4/\ell^2 + 2r_+^2(1 + a^2/\ell^2) + a^2}. \quad (4.39)$$

³The volume of S^5 is π^3 .

The divergences at radial infinity in the bulk action are exactly canceled by the ones in the Euclidean boundary term, such that we get the finite result

$$I_7^E = -\frac{\beta\pi^2\ell^4}{1280\Xi G} \left[160 \left(\frac{r_+^4 a^2}{\ell^6} + \frac{r_+^6}{\ell^6} - \frac{m}{\ell^4} \right) + \frac{a^6}{\ell^6} + 5\frac{a^4}{\ell^4} + 50\Xi \right], \quad (4.40)$$

that can be conveniently rewritten as

$$I_7^E = \beta E_0 + I_7'^E, \quad (4.41)$$

where

$$I_7'^E = \frac{\beta\pi^2}{8G\ell^2\Xi} (m\ell^2 - r_+^4(r_+^2 + a^2)) \quad (4.42)$$

corresponds to the value of the Euclidean action computed in a background-substraction method [30]. Eq. (4.42) satisfies the thermodynamical relation

$$S = \beta(E - \Omega J) - I_7'^E = \frac{Area}{4G}, \quad (4.43)$$

for the energy (4.37), angular momentum (4.35), the angular velocity respect to a non-rotating frame at infinity

$$\Omega = \frac{a(1 + r_+^2\ell^{-2})}{r_+^2 + a^2}, \quad (4.44)$$

and the area of the event horizon

$$Area = \pi^3 \frac{r_+^3(r_+^2 + a^2)}{\Xi}. \quad (4.45)$$

In turn, the Euclidean action (4.41), computed using the background-independent Kounterterms prescription, obeys

$$S = \beta(\mathcal{E} - \Omega J) - I_7^E = \frac{Area}{4G}, \quad (4.46)$$

for a thermodynamical energy consistently shifted in the vacuum energy $\mathcal{E} = E + E_0$.

5. General odd-dimensional case

We have already introduced the general form the Kounterterms series adopts in any odd dimension $D = 2n + 1$, eqs. (2.15)–(2.17). The parametric integrations provide the relative coefficients of the boundary terms when eq. (2.17) is expanded as a polynomial in the extrinsic and intrinsic curvature

$$B_{2n} = n! \sqrt{-h} \sum_{p=0}^{n-1} \frac{(2n - 2p - 3)!!}{\ell^{2(n-1-p)}} b_{2n}^{(p)}, \quad (5.1)$$

where

$$b_{2n}^{(p)} = \delta_{[j_1 \dots j_{2p+1}]^{[i_1 \dots i_{2p+1}]}} \sum_{q=0}^p \frac{(-1)^{p-q}}{(p-q)!q!} \frac{2^{n-(p+q+1)}}{(n-q)} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2q-1} i_{2q}}^{j_{2q-1} j_{2q}} K_{i_{2q+1}}^{j_{2q+1}} \dots K_{i_{2p+1}}^{j_{2p+1}}. \quad (5.2)$$

The surface term obtained from an arbitrary variation of the action (2.1)–or equivalently, (2.18)– has a more involved form in the general odd-dimensional case. In the appendix D, we summarize the process of variation of the action in the general case, cast in an appropriate form that allows us to impose the asymptotic conditions for AAdS spacetimes discussed above

$$\begin{aligned}
 \delta I_{2n+1} = & -2 \int_{\partial M} \epsilon_{a_1 \dots a_{2n}} \delta K^{a_1} e^{a_2} \left[\kappa_D e^{a_3} \dots e^{a_{2n}} + n c_{2n} \int_0^1 dt \left(\hat{R}^{a_3 a_4} + \frac{t^2}{\ell^2} e^{a_3} e^{a_4} \right) \times \dots \right. \\
 & \left. \times \left(\hat{R}^{a_{2n-1} a_{2n}} + \frac{t^2}{\ell^2} e^{a_{2n-1}} e^{a_{2n}} \right) \right] \\
 & - n c_{2n} \int_0^1 dt t \epsilon_{a_1 \dots a_{2n}} (\delta K^{a_1} e^{a_2} - K^{a_1} \delta e^{a_2}) \left(R^{a_3 a_4} - t^2 K^{a_3} K^{a_4} + \frac{t^2}{\ell^2} e^{a_3} e^{a_4} \right) \times \dots \\
 & \times \left(R^{a_{2n-1} a_{2n}} - t^2 K^{a_{2n-1}} K^{a_{2n}} + \frac{t^2}{\ell^2} e^{a_{2n-1}} e^{a_{2n}} \right). \quad (5.3)
 \end{aligned}$$

As we have already seen in the five and seven-dimensional cases, no matter the terms that multiply the curl $\epsilon_{a_1 a_2 \dots a_{2n}} (\delta K^{a_1} e^{a_2} - K^{a_1} \delta e^{a_2})$, the asymptotic conditions (making the full action stationary then reduces to fix the coupling constant c_{2n} of the boundary term. This can be done demanding the spacetime to be of constant curvature at the asymptotic region (eq. (3.10)). In a Riemannian manifold, this condition is equivalent to asymptotic flatness for the curvature of the AdS group

$$F = dA + A \wedge A = \frac{1}{2} \left(\hat{R}^{AB} + \frac{e^A e^B}{\ell^2} \right) J_{AB} + \frac{T^A}{\ell} P_A, \quad (5.4)$$

where $A = \frac{1}{2} \omega^{AB} J_{AB} + \frac{e^A}{\ell} P_A$ is the $SO(D-1, 2)$ group connection field, $\{J_{AB}, P_A\}$ are the generators of AdS rotations and translations, respectively, and $T^A = \frac{1}{2} T_{\mu\nu}^A dx^\mu \wedge dx^\nu$ is the two-form torsion.

Assuming (3.10), we obtain the value

$$c_{2n} = -\ell^{2n-2} \frac{\kappa_D}{n} \left[\int_0^1 dt (t^2 - 1)^{n-1} \right]^{-1}, \quad (5.5)$$

$$= -\frac{(-\ell^2)^{n-1}}{4^{n+1} \pi G n [(n-1)!]^2} \quad (5.6)$$

whose fixing is equivalent to canceling the highest-order divergences in the Euclidean action.

5.1 Noether charges

The conserved current associated to an isometry and prescribed by the Noether theorem, in the general odd-dimensional case is

$$*J = -\Theta(e^a, K^a, \delta e^a, \delta K^a) - i_\xi (L_{2n+1} + c_{2n} dB_{2n}). \quad (5.7)$$

Here, L_{2n+1} is the bulk Lagrangian in $2n+1$ dimensions, B_{2n} is the regularizing Kountert-terms series and Θ is the surface term in the variation of the action (5.3).

Either replacing the explicit form of the terms in eq. (5.7) to write down the Noether current as $*J = dQ(\xi)$ or employing the shortcut to the charge derivation in appendix C

$$Q(\xi) = \int_{\partial\Sigma} 2\kappa_D \varepsilon_{a_1 \dots a_{2n}} i_\xi K^{a_1} e^{a_2} \dots e^{a_{2n}} + c_{2n} \left(i_\xi K^a \frac{\delta B_{2n}}{\delta K^a} + i_\xi e^a \frac{\delta B_{2n}}{\delta e^a} \right) \quad (5.8)$$

the conserved charge is split as

$$Q(\xi) = q(\xi) + q_0(\xi). \quad (5.9)$$

The first contribution can be traced back to the first two lines in the surface term (5.3)

$$q(\xi) = \frac{1}{2^{n-2}} \int_{\partial\Sigma} \kappa_D \sqrt{-h} \varepsilon_{i_1 \dots i_{2n}} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left[\delta_{[j_3 j_4]}^{[i_3 i_4]} \dots \delta_{[j_{2n-1} j_{2n}]}^{[i_{2n-1} i_{2n}]} + \right. \\ \left. + n c_{2n} \int_0^1 dt \left(\hat{R}_{j_3 j_4}^{i_3 i_4} + \frac{t^2}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \dots \left(\hat{R}_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} + \frac{t^2}{\ell^2} \delta_{[j_{2n-1} j_{2n}]}^{[i_{2n-1} i_{2n}]} \right) \right] dx^{j_2} \dots dx^{j_{2n}} \quad (5.10)$$

and the second one comes from the curl term in the same term

$$q_0(\xi) = -\frac{n c_{2n}}{2^{n-2}} \int_{\partial\Sigma} \sqrt{-h} \int_0^1 dt t \varepsilon_{i_1 i_2 \dots i_{2n}} \xi^k (\delta_{j_2}^{i_2} K_k^{i_1} + \delta_k^{i_2} K_{j_2}^{i_1}) \left(R_{j_3 j_4}^{i_3 i_4} - t^2 K_{[j_3 j_4]}^{[i_3 i_4]} + \frac{t^2}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \dots \\ \left(R_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} - t^2 K_{[j_{2n-1} j_{2n}]}^{[i_{2n-1} i_{2n}]} + \frac{t^2}{\ell^2} \delta_{[j_{2n-1} j_{2n}]}^{[i_{2n-1} i_{2n}]} \right) dx^{j_2} dx^{j_3} \dots dx^{j_{2n}}. \quad (5.11)$$

Evaluating the formula (5.10) for $\xi = \partial_t$ in AAdS static black holes (3.22)–(3.23) gives

$$q(\partial_t) = (D-2)! \text{Vol}(\Sigma_{D-2}^k) \lim_{r \rightarrow \infty} (\Delta^2)' r \left\{ \kappa_D r^{2(n-1)} + n c_{2n} \int_0^1 dt \left(k - \Delta^2 + \frac{t^2 r^2}{\ell^2} \right)^{n-1} \right\}, \quad (5.12)$$

where the derivative of the function in the metric is

$$(\Delta^2)' r = 2(D-3) \frac{\omega_D G \mu}{r^{D-3}} + 2 \frac{r^2}{\ell^2}. \quad (5.13)$$

Expanding the second term as

$$n c_{2n} \int_0^1 dt \left(k - \Delta^2 + \frac{t^2 r^2}{\ell^2} \right)^{n-1} = \kappa_D \left(-r^{2(n-1)} + (2n-1) \omega_D G \frac{\ell^2}{r^2} \mu + \dots \right) \quad (5.14)$$

where the additional contributions of lower order in r are irrelevant in the limit $r \rightarrow \infty$, the topological black hole mass is finally

$$q(\partial_t) = M = \frac{\text{Vol}(\Sigma_{D-2}^k)}{\text{Vol}(S^{D-2})} \mu.$$

The other formula in the conserved charge, $q_0(\xi)$, specialized for a timelike Killing vector and topological black holes, produces

$$q_0(\partial_t) = -2 c_{2n} (D-2)! \text{Vol}(\Sigma_{D-2}^k) \lim_{r \rightarrow \infty} \left(\Delta^2 - \frac{r(\Delta^2)'}{2} \right) \int_0^1 dt t \left(k - t^2 \Delta^2 + \frac{t^2 r^2}{\ell^2} \right)^{n-1}, \quad (5.15)$$

where the explicit evaluation of

$$\left(\Delta^2 - \frac{r(\Delta^2)'}{2}\right) = k - (D-1)\frac{\omega_D G}{r^{D-3}}\mu, \quad (5.16)$$

$$\left(k - t^2\Delta^2 + \frac{t^2 r^2}{\ell^2}\right) = k(1-t^2) + 2t^2\frac{\omega_D G}{r^{D-3}}\mu, \quad (5.17)$$

introduces at most finite contributions to the zero-point (vacuum) energy

$$q_0(\partial_t) = E_0 = (-k)^n \frac{Vol(\Sigma_{D-2}^k)}{8\pi G} \ell^{2n-2} \frac{(2n-1)!!^2}{(2n)!}. \quad (5.18)$$

This expression corroborates the general formula for the vacuum energy for odd-dimensional AdS spacetime conjectured in ref. [8], based on an extrapolation of explicit results in the counterterms method up to nine dimensions.

In order to verify the consistency of the black hole thermodynamics, we compute the total Euclidean action

$$I_{2n+1}^E = I_{\text{bulk}}^E + c_{2n} \int_{\partial M} B_{2n}^E, \quad (5.19)$$

for SAdS black hole. The bulk term is a total derivative, such that the integration in the radial coordinate in the interval $[r_+, \infty)$ is simply

$$I_{\text{bulk}}^E = -\kappa_D (D-2)! Vol(\Sigma_{D-2}^k) \beta \{(\Delta^2)' r^{D-2}\}|_{r_+}^{\infty}, \quad (5.20)$$

and the Euclidean boundary term is

$$\int_{\partial M} B_{2n}^E = 2(D-2)! Vol(\Sigma_{D-2}^k) \beta \left[\frac{r(\Delta^2)'}{2} \int_0^1 dt \left(k - \Delta^2 + \frac{t^2 r^2}{\ell^2}\right)^{n-1} + \right. \quad (5.21)$$

$$\left. + \left(\Delta^2 - \frac{r(\Delta^2)'}{2}\right) \int_0^1 dt t \left(k - t^2\Delta^2 + \frac{t^2 r^2}{\ell^2}\right)^{n-1} \right] \Big|_{r_+}^{\infty}. \quad (5.22)$$

The total contribution at $r = \infty$ can be identify as $-\beta(M + E_0)$ and then the total Euclidean action is

$$I_{2n+1}^E = \frac{Vol(\Sigma_{D-2}^k)}{16\pi G} \beta r_+^{D-2} (\Delta^2)'|_{r_+} - \beta(M + E_0). \quad (5.23)$$

The definition of thermodynamic energy \mathcal{E} in eq. (3.37) recovers the total energy in the Noether theorem $Q(\partial_t) = q(\partial_t) + q_0(\partial_t)$, such that eq. (3.33) implies an entropy for SAdS black hole

$$S = \frac{Vol(\Sigma_{D-2}^k) r_+^{D-2}}{4G} = \frac{Area}{4G}. \quad (5.24)$$

6. Conclusions

In this paper, we have explicitly shown the tensorial form of the Kounterterms that regularize the action for AdS gravity in all odd dimensions.

The key point of the construction is a well-principle action principle that respects boundary conditions consistent with the asymptotic behavior of a generic AAdS spacetime.

A definite form of the boundary terms achieves a finite action principle: the action is stationary under arbitrary variations of the fields and the conserved charges and the Euclidean action are finite.

In the general odd-dimensional case, we have not computed the conserved quantities in AAdS solutions other than for topological SAdS black holes.

Further evaluation of more complex solutions will be certainly more involved, but it could also reveal the general form that the vacuum energy adopts for certain AAdS spaces. Particularly interesting could be extending the results of the vacuum energy in refs. [29, 12] to the recently generalized multi-parameter Kerr-AdS black hole [30]. It is also worthwhile to notice that the existing results on vacuum energy for Kerr-AdS solutions, reproduced here, have been critically revised in ref. rotation parameters. There, it is claimed that, in the same way the energy and the angular velocities are referred to a coordinate frame that is nonrotating at infinity, one should choose the Einstein static universe as the metric on the conformal boundary of the rotating black hole. As a consequence, the corresponding Casimir energy is genuinely a constant and matches the one for SAdS black hole.

But, how can we be sure that $q_0(\partial_t)$ would always produce the vacuum energy?

The reasoning is quite simple and has to do with the general form of the other quantity that enters in the Noether charge, $q(\xi)$. It can be proved that eq. (5.10) can be factorized by the l.h.s. of eq. (1.4), in the corresponding boundary indices

$$q(\xi) = \frac{nc_{2n}}{2^{n-2}} \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 \dots i_{2n}} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left(\hat{R}_{j_3 j_4}^{i_3 i_4} + \frac{1}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \mathcal{P}_{j_5 \dots j_{2n}}^{i_5 \dots i_{2n}} dx^{j_2} \dots dx^{j_{2n}}, \quad (6.1)$$

where \mathcal{P} is a Lovelock-type polynomial of $(n - 2)$ degree in the Riemann tensor \hat{R}_{kl}^{ij} and the antisymmetrized Kronecker delta $\delta_{[kl]}^{[ij]}$

$$\mathcal{P}_{j_5 \dots j_{2n}}^{i_5 \dots i_{2n}} = \sum_{p=0}^{n-2} \frac{D_p}{\ell^{2p}} \hat{R}_{j_5 j_6}^{i_5 i_6} \dots \hat{R}_{j_{2(n-p)-1} j_{2(n-p)}}^{i_{2(n-p)-1} i_{2(n-p)}} \delta_{[j_{2(n-p)+1} j_{2(n-p)+1}]^{[i_{2(n-p)+1} i_{2(n-p)+1}]} \dots \delta_{[j_{2n-1} j_{2n}]^{[i_{2n-1} i_{2n}]}, \quad (6.2)$$

with the coefficients of the expansion given by

$$D_p = \sum_{q=0}^p \frac{(-1)^{p-q}}{2q+1} \binom{n-1}{q}. \quad (6.3)$$

Logically, using the identities for the antisymmetrized Kronecker deltas, one could express \mathcal{P} in terms of the Riemann tensor only, but prefer the above form to make easier the connection with the explicit cases developed so far. As an example, in nine dimensions, the charge $q(\xi)$ is

$$q(\xi) = c_8 \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 \dots i_8} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left(\hat{R}_{j_3 j_4}^{i_3 i_4} + \frac{1}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \left[\hat{R}_{j_5 j_6}^{i_5 i_6} \hat{R}_{j_7 j_8}^{i_7 i_8} + \frac{3}{5\ell^4} \delta_{[j_5 j_6]}^{[i_5 i_6]} \delta_{[j_7 j_8]}^{[i_7 i_8]} \right] dx^{j_2} \dots dx^{j_8} \quad (6.4)$$

whereas in eleven dimensions

$$\begin{aligned}
 q(\xi) = & \frac{5c_{10}}{8} \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 \dots i_{10}} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left(\hat{R}_{j_3 j_4}^{i_3 i_4} + \frac{1}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \left[\hat{R}_{j_5 j_6}^{i_5 i_6} \hat{R}_{j_7 j_8}^{i_7 i_8} \hat{R}_{j_9 j_{10}}^{i_9 i_{10}} + \right. \\
 & \left. + \frac{1}{3\ell^2} \hat{R}_{j_5 j_6}^{i_5 i_6} \hat{R}_{j_7 j_8}^{i_7 i_8} \delta_{[j_9 j_{10}]}^{[i_9 i_{10}]} + \frac{13}{15\ell^4} \hat{R}_{j_5 j_6}^{i_5 i_6} \delta_{[j_7 j_8]}^{[i_7 i_8]} \delta_{[j_9 j_{10}]}^{[i_9 i_{10}]} - \frac{31}{105\ell^6} \delta_{[j_5 j_6]}^{[i_5 i_6]} \delta_{[j_7 j_8]}^{[i_7 i_8]} \delta_{[j_9 j_{10}]}^{[i_9 i_{10}]} \right] dx^{j_2} \dots dx^{j_{10}}.
 \end{aligned}
 \tag{6.5}$$

As a consequence, $q(\xi)$ always vanishes for a spacetime that is globally AdS. This argument indicates that $q_0(\xi)$ in eq. (5.11) for a timelike Killing vector is indeed a covariant formula for the vacuum energy in AAdS spacetimes.

The expression for the vacuum energy has been also recognized as the action of a Killing vector in the Euclidean continuation of the boundary term B_{2n} for explicit black hole solutions. It is expected that a generic thermodynamical relation $S = \beta(E + E_0 - \sum_i \Omega_i J_i) - I_{2n+1}^E = Area/4G$ holds for any AAdS spacetime that accepts a timelike and a set of rotational Killing vectors. Carrying out a similar procedure as Wald's formalism [43], expressions for E and E_0 should be mapped exactly to contributions from the bulk and the boundary after acting with a global isometry $\xi = \partial_t + \Omega_i^\infty \partial_{\phi_i}$ on them, where the angular velocities at infinity are given by $\Omega_i^\infty = a_i^2/\ell^2$.

In odd-dimensional spacetimes, the regularized action is not invariant under the full AdS group. Radial bulk diffeomorphisms, which generate a Weyl transformation on the boundary, are generically broken by the conformal anomaly. In the standard counterterms approach, this is reflected in a nonvanishing trace of the regularized stress tensor [3, 44, 45].

In the present framework, the surface term from an arbitrary variation of the action contains also variations of the extrinsic curvature (usually canceled by the Gibbons-Hawking term), such that a boundary stress tensor definition is not straightforward. The answer to this point should come from direct comparison of the Kounterterms series with Dirichlet counterterms, e.g., by expansion of the tensorial quantities in FG form. For instance, in this way, it can be proved that in three-dimensional AdS gravity, the Kounterterms prescription reduces to the Dirichlet regularization up to a topological invariant on the boundary [18]. The matching of the results in this paper with the ones obtained by standard holographic renormalization indicates that both procedures could also be equivalent in higher dimensions.

If a relation of the Noether charges to a regularized stress tensor τ_{ij} is possible, we might expect that a similar splitting as (5.9) would also appear on it. At a more speculative level, such identification could reveal a connection between the part of τ_{ij} that generates the vacuum energy and the one that produces the Weyl anomaly, what has not yet been understood in the standard holographic renormalization.

Could the term B_d represent the full counterterms series? We do not know yet the answer to this question. One could only argue that is very unlikely that other boundary terms can be added on top of the Kounterterms series, that still preserve the AAdS boundary conditions extensively used here. This argument might free this procedure from the ambiguities of the Dirichlet regularization [46].

In any case, what is particularly appealing in this formulation is the relation of Kounterterms to topological invariants in $D = 2n$ [15] and to Chern-Simons-like forms (transgression forms) in $D = 2n+1$ [47], which might provide some further insight on the problem of regularization of Einstein-Hilbert-AdS gravity, but also in Einstein-Gauss-Bonnet [24] and other theories with higher curvature terms (see, e.g., [17, 52]).

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A. Invariant polynomials, Chern-Simons and transgression forms

In this appendix we review the notion of transgression form as the natural extension of a Chern-Simons density that restores gauge invariance through the introduction of an additional gauge connection [48–51].

Let us consider in $2n + 2$ dimensions an invariant polynomial $P(F)$ of the form

$$P(F) = \langle F^{n+1} \rangle \tag{A.1}$$

where $F = \frac{1}{2}F_{\mu\nu}^I T_I dx^\mu dx^\nu = dA + A \wedge A$ is the curvature two-form associated to the gauge potential $A = A_\mu^I T_I dx^\mu$ of the Lie group G , with a generators set $\{T_I\}$. The symbol $\langle \dots \rangle$ stands for a totally symmetric invariant trace of the generators in the adjoint representation of G

$$\langle T_{I_1} \dots T_{I_{n+1}} \rangle = g_{I_1 \dots I_{n+1}}. \tag{A.2}$$

The invariant polynomial (A.1) is a closed form

$$dP(F) = 0 \tag{A.3}$$

and therefore, by virtue of the Poincaré lemma, locally exact

$$P(F) = d\mathcal{C}_{2n+1}(A, F) \tag{A.4}$$

what provides the definition of a Chern-Simons density as the integration over a continuous parameter u

$$\mathcal{C}_{2n+1}(A, F) \equiv (n + 1) \int_0^1 du \langle AF_u^n \rangle \tag{A.5}$$

with $A_u = uA$ and $F_u = dA_u + A_u^2$.

A similar relation defines a transgression form $\mathcal{T}_{2n+1}(A, \bar{A})$, that involves two gauge potentials A and \bar{A} in the same homotopy class, with curvatures F and \bar{F} , respectively

$$\langle F^{n+1} \rangle - \langle \bar{F}^{n+1} \rangle = d\mathcal{T}_{2n+1}(A, \bar{A}). \tag{A.6}$$

The explicit formula for the transgression form is also given by a parametric integration

$$\mathcal{T}_{2n+1}(A, \bar{A}) \equiv (n+1) \int_0^1 dt \langle (A - \bar{A}) F_t^n \rangle, \quad (\text{A.7})$$

where $F_t = dA_t + A_t^2$ is the curvature associated to the interpolating gauge connection $A_t = tA + (1-t)\bar{A}$. On the contrary to Chern-Simons densities, transgression forms are truly invariant under finite gauge transformations in the group G .

The explicit formula for (A.7) is a consequence of the use of the *Cartan homotopy operator* k_{01} , which acts on generic polynomials $P(F_t, A_t)$ and is defined as

$$k_{01}P(F_t, A_t) = \int_0^1 dt l_t P(F_t, A_t), \quad (\text{A.8})$$

where the action of the operator l_t on arbitrary polynomials of A_t and F_t can be worked out from the relations

$$l_t A_t = 0, \quad l_t F_t = A - \bar{A}. \quad (\text{A.9})$$

The operator l_t acts as an antiderivative $l_t(\Lambda_p \Sigma_q) = (l_t \Lambda_p) \Sigma_q + (-1)^p \Lambda_p (l_t \Sigma_q)$, where Λ_p and Σ_q are p and q -forms, respectively.

It is particularly useful to express the curvature F_t as

$$F_t = \bar{F} + t\bar{D}(A - \bar{A}) + t^2(A - \bar{A})^2, \quad (\text{A.10})$$

with the curvature \bar{F} associated to the connection \bar{A} ($\bar{F} = d\bar{A} + \bar{A}^2$) and the covariant derivative in \bar{A} given by $\bar{D}(A - \bar{A}) = d(A - \bar{A}) + \bar{A}(A - \bar{A}) + (A - \bar{A})\bar{A}$.

In four dimensions, one can re-obtain the formula for the second Chern form (2.12) from the generic transgression formula taking two gauge connections for the Lorentz group $\text{SO}(3, 1)$, that is, $A = \frac{1}{2}\omega^{AB} J_{AB}$ and $\bar{A} = \frac{1}{2}\bar{\omega}^{AB} J_{AB}$ and the invariant tensor for the Lorentz generators $\{J_{AB} J_{CD}\} = \varepsilon_{ABCD}$. In this way, the interpolating connection in terms of the Second Fundamental Form (2.6) is

$$A_t = \frac{1}{2}\omega_t^{AB} J_{AB} = \frac{1}{2}(\bar{\omega}^{AB} + t\theta^{AB})J_{AB}, \quad (\text{A.11})$$

and its corresponding curvature

$$F_t = \frac{1}{2}\hat{R}_t^{AB} J_{AB} = \frac{1}{2} \left[\hat{R}^{AB} + t\bar{D}\theta^{AB} + t^2\theta_C^A \theta^{CB} \right] J_{AB}, \quad (\text{A.12})$$

where \hat{R}^{AB} and \bar{D} are the curvature and the covariant derivative in the spin connection $\bar{\omega}^{AB}$. When plugged in eqs. (A.6), (A.7), the transgression form for the Lorentz group satisfies the local relation

$$\mathcal{E}_4(\hat{R}) - \mathcal{E}_4(\bar{R}) = 2d \left(\int_0^1 dt \varepsilon_{ABCD} \theta^{AB} \left(\hat{R}^{CD} + t\bar{D}\theta^{CD} + t^2\theta_F^C \theta^{FD} \right) \right), \quad (\text{A.13})$$

where $\mathcal{E}_4 = \varepsilon_{ABCD} \hat{R}^{AB} \hat{R}^{CD}$ is the Euler-Gauss-Bonnet topological invariant. For a radial foliation of the spacetime (2.10), an adequate choice of the reference spin connection corresponds to the matching conditions (2.11) for the Second Fundamental Form and relates the

components \hat{R}^{ab} to the intrinsic curvature at the boundary, i.e., $\hat{R}^{ab} = R^{ab}(h)$. In doing so, the second Euler term in the l.h.s. of eq. (A.13) vanishes identically, so does the second term in the r.h.s.

Global considerations show that the Euler term and the second Chern form B_3 (2.12) in four dimensions are equivalent up a topological number (Euler characteristic $\chi(M_4)$)

$$\int_{M_4} \mathcal{E}_4(\hat{R}) = 32\pi^2 \chi(M_4) + \int_{\partial M_4} B_3, \tag{A.14}$$

revealing the profound connection of the Kounterterms method with topological invariants.

In a similar fashion, the higher even-dimensional Kounterterms B_{2n-1} (2.15) are related to the corresponding Euler term in $D = 2n$ dimensions by virtue of the Euler theorem

$$\int_{M_{2n}} \mathcal{E}_{2n}(\hat{R}) = (-4\pi)^n n! \chi(M_{2n}) + \int_{\partial M_{2n}} B_{2n-1}. \tag{A.15}$$

For any invariant polynomial $P(F_t, A_t)$, it can be verified that

$$(l_t d + dl_t)P(F_t, A_t) = \frac{\partial}{\partial t}P(F_t, A_t) \tag{A.16}$$

which, when integrated between 0 and 1, recovers the Cartan homotopy formula

$$(k_{01}d + dk_{01})P(F_t, A_t) = P(F, A) - P(\bar{F}, \bar{A}). \tag{A.17}$$

In particular, for $P = \mathcal{C}_{2n+1}$, the above relation allows us to express a transgression form as the difference of two Chern-Simons densities plus a boundary term

$$\mathcal{T}_{2n+1} = \mathcal{C}_{2n+1}(A, F) - \mathcal{C}_{2n+1}(\bar{A}, \bar{F}) + d\Xi_{2n}(A, F; \bar{A}, \bar{F}). \tag{A.18}$$

The $2n$ -form Ξ_{2n} is defined by the action of the Cartan homotopy operator on a Chern-Simons term

$$\Xi_{2n}(A, F; \bar{A}, \bar{F}) \equiv k_{01}\mathcal{C}_{2n+1} \tag{A.19}$$

whose explicit form is given by

$$\Xi_{2n} = n(n+1) \int_0^1 ds \int_0^1 dt s \langle A_t(A - \bar{A}) F_{st}^{n-1} \rangle \tag{A.20}$$

where $F_{st} = sF_t + s(s-1)A_t^2$.

When the gauge group is AdS, the connection field is written as

$$A = \frac{1}{2}\omega^{AB}J_{AB} + e^A P_A, \tag{A.21}$$

where J_{AB} and P_A are the generators of rotations and AdS translations, respectively. For the trace of the generators of $\text{SO}(2n, 2)$, we take $\langle J_{A_1 A_2}, \dots, J_{A_{2n-1} A_{2n}}, P_{A_{2n+1}} \rangle =$

$\frac{2^n}{(n+1)}\varepsilon_{A_1\dots A_{2n+1}}$, such that each Chern-Simons form in (A.18) can be written as a bulk Lovelock Lagrangian (a polynomial in the curvature and the vielbein)

$$\begin{aligned} \mathcal{C}_{2n+1}(A, F) &= \mathcal{L}_{CS-AdS}(\hat{R}, e) \\ &= \int_0^1 dt \varepsilon_{A_1\dots A_{2n+1}} (\hat{R}^{A_1 A_2} + \frac{t^2}{\ell^2} e^{A_1} e^{A_2}) \dots (\hat{R}^{A_{2n-1} A_{2n}} + \frac{t^2}{\ell^2} e^{A_{2n-1}} e^{A_{2n}}) e^{A_{2n+1}} \end{aligned} \quad (\text{A.22})$$

plus a surface term whose explicit form is involved and has to be worked out case by case. However, the matching conditions for the Second Fundamental Form (2.11) –that single out the boundary correction in the Euler theorem– and the condition $\bar{e}^A = 0$ (in order to avoid a background-dependent bimetric formulation) replaced in the transgression form (A.18) result into a gravity action

$$I_{CS-AdS} = \int_{M_{2n+1}} \mathcal{T}_{2n+1} = \int_{M_{2n+1}} \mathcal{L}_{CS-AdS}(\hat{R}, e) + \int_{\partial M_{2n+1}} B_{2n}(\theta, e) \quad (\text{A.23})$$

with a boundary term $B_{2n}(\theta, e)$ given by the formula (2.15). The above total action is regularized both in its Euclidean continuation and in the Noether charges. It is quite remarkable that the contributions from the bulk terms in (A.18) combine to the surface term (A.20) to produce a compact expression for the boundary terms in this particular Lovelock theory as the double integral in eq. (2.15). But it is even more surprising that the same $2n$ -form is useful to regularize other gravity theories, including Einstein-Hilbert-AdS, what is shown explicitly in this paper.

B. Useful identities

Let us consider the five-dimensional Kounterterms as an example of the equivalence between differential forms and tensorial notation

$$\begin{aligned} B_4 &= \varepsilon_{A_1\dots A_5} \theta^{A_1 A_2} e^{A_3} \left(R^{A_4 A_5} + \frac{1}{2} \theta^A{}_C \theta^{C A_5} + \frac{1}{6\ell^2} e^{A_4} e^{A_5} \right), \\ &= 2\varepsilon_{1a_1 a_2 a_3 a_4} \theta^{1a_1} e^{a_2} \left(R^{a_3 a_4} + \frac{1}{2} \theta^{a_3}{}_1 \theta^{1a_4} + \frac{1}{6\ell^2} e^{a_3} e^{a_4} \right), \\ &= -2\varepsilon_{a_1 a_2 a_3 a_4} K^b{}_c \left(R^{a_3 a_4} - \frac{1}{2} K^{a_3} K^{a_4} + \frac{1}{6\ell^2} e^{a_3} e^{a_4} \right), \\ &= -\varepsilon_{a_1 a_2 a_3 a_4} e_{j_1}^{a_1} e_{j_2}^{a_2} e_{j_3}^{a_3} e_{j_4}^{a_4} K_{i_1}^{j_1} \delta_{i_2}^{j_2} \left(R_{i_3 i_4}^{j_3 j_4} - \frac{1}{2} K_{[i_3 i_4]}^{[j_3 j_4]} + \frac{1}{6\ell^2} \delta_{[i_3 i_4]}^{[j_3 j_4]} \right) dx^{i_1} \wedge \dots \wedge dx^{i_4}, \\ &= \sqrt{-h} \delta_{[j_1 j_2 j_3]}^{[i_1 i_2 i_3]} K_{i_1}^{j_1} \left(R_{i_2 i_3}^{j_2 j_3} - K_{i_2}^{j_2} K_{i_3}^{j_3} + \frac{1}{3\ell^2} \delta_{i_2}^{j_2} \delta_{i_3}^{j_3} \right) d^4 x, \end{aligned} \quad (\text{B.1})$$

where we have used the identities

$$\varepsilon_{a_1\dots a_{2n}} e_{j_1}^{a_1} \dots e_{j_{2n}}^{a_{2n}} = -\sqrt{-h} \varepsilon_{j_1\dots j_{2n}}, \quad (\text{B.2})$$

with the definition $\sqrt{-h} = \det(e)$, the volume element

$$dx^{i_1} \wedge \dots \wedge dx^{i_{2n}} = \varepsilon^{i_1\dots i_{2n}} d^{2n} x, \quad (\text{B.3})$$

and the general property for antisymmetrized Kronecker deltas

$$\delta_{[j_1 \dots j_p]}^{[i_1 \dots i_p]} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_m}^{j_m} = \frac{(r-p+m)!}{(r-p)!} \delta_{[j_{m+1} \dots j_p]}^{[i_{m+1} \dots i_p]} \quad (\text{B.4})$$

where $p > m$ and r as the range of the indices.

C. Noether's theorem

We now recall the standard construction of the conserved quantities associated to asymptotic symmetries of the action through the Noether's theorem.

Let us consider an action that is the integral of a D -form Lagrangian density in D dimensions

$$L = \frac{1}{D!} L_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D}. \quad (\text{C.1})$$

An arbitrary variation $\bar{\delta}$ acting on the fields can be always decomposed in a functional variation δ plus the variation due to an infinitesimal change in the coordinates $x'^{\mu} = x^{\mu} + \eta^{\mu}$. For a p -form field φ , the latter variation is given by the Lie derivative $\mathcal{L}_{\eta}\varphi$ along the vector η^{μ} , that can be written as $\mathcal{L}_{\eta}\varphi = (dI_{\eta} + I_{\eta}d)\varphi$, where d is the exterior derivative and I_{η} is the contraction operator.⁴ The functional variation δ acting on L produces the equations of motion plus a surface term $\Theta(\varphi, \delta\varphi)$. The Lie derivative contributes only with another surface term because $dL = 0$ in D dimensions.

The Noether's theorem provides a conserved current associated to the invariance under diffeomorphisms of the Lagrangian L , that is given by [23, 53]

$$*J = -\Theta(\varphi, \delta\varphi) - i_{\eta}L. \quad (\text{C.2})$$

In case that the diffeomorphism ξ is a Killing vector, we have $\delta\varphi = -\mathcal{L}_{\xi}\varphi$, with \mathcal{L}_{ξ} the Lie derivative along the vector ξ^{μ} . Because the surface term Θ is linear in the variations of the fields, the current takes the form

$$*J = \Theta(\varphi, \mathcal{L}_{\xi}\varphi) - i_{\xi}L. \quad (\text{C.3})$$

(see also [54] and, for a recent discussion, [55, 56]).

As the current is conserved ($d*J = 0$), $*J$ can always be written locally as an exterior derivative of a quantity. In general, the boundary ∂M consists of two spacelike surfaces (at initial time Σ_{t_1} and at final time Σ_{t_2}) and a timelike surface Σ_{∞} (at spatial infinity). Only when the current can be written globally as an exact form $*J = dQ(\xi)$, we can integrate the charge $Q(\xi)$ in a $(D-2)$ -dimensional surface $\partial\Sigma$ (a constant-time slice in Σ_{∞} , as we assume flux conservation through Σ_{t_1} and Σ_{t_2}).

Let us consider now a Lagrangian \mathcal{L} that differs from L in a boundary term $d\beta$

$$\mathcal{L} = L + d\beta, \quad (\text{C.4})$$

⁴The action of the contraction operator I_{η} over a p -form $\alpha_p = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$ is given by $I_{\eta}\alpha_p = \frac{1}{(p-1)!} \eta^{\nu} \alpha_{\nu\mu_1 \dots \mu_{p-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p-1}}$

so that the conserved current is modified as

$$*\mathcal{J} = \Theta(\varphi, \mathcal{L}_\xi \varphi) - i_\xi L + \frac{\delta B}{\delta \varphi} \mathcal{L}_\xi \varphi - i_\xi dB \tag{C.5}$$

$$= d(Q(\xi) + i_\xi B). \tag{C.6}$$

The above formula provides a useful shortcut to find the conserved charges of a Lagrangian supplemented by a boundary term as

$$Q(\xi) = Q(\xi) + i_\xi B. \tag{C.7}$$

D. Variation of B_{2n}

The variation of the Kounterterms series B_{2n} has an expanded form given by

$$\begin{aligned} \delta B_{2n} = & -2n \int_0^1 dt \varepsilon \sum_{k=0}^{n-1} C_k^{n-1} \sum_{l=0}^{n-1-k} C_l^{n-1-k} R^{n-1-k-l} \int_0^t dst^{2k+1} (-1)^k \delta K^{2k+1} s^{2l} e^{2l+1} \\ & -2n \int_0^1 dt \varepsilon \sum_{k=0}^{n-1} C_k^{n-1} \sum_{l=0}^{n-1-k} C_l^{n-1-k} R^{n-1-k-l} \int_0^t dst^{2l+1} (-1)^l K^{2l+1} s^{2k} \delta e^{2k+1}, \end{aligned} \tag{D.1}$$

where variations of the intrinsic curvature produce surface terms that are identically vanishing on the boundary. After some algebraic manipulations, we have

$$\begin{aligned} \delta B_{2n} = & -2n \int_0^1 dt \varepsilon \delta K e \sum_{k=0}^{n-1} C_k^{n-1} (R + t^2 ee)^{n-1-k} (-KK)^k (1 - t^{2k+1}) \\ & -2n \int_0^1 dt \varepsilon K \delta e \sum_{k=0}^{n-1} C_k^{n-1} (R + t^2 e^2)^{n-1-k} t^{2k+1} (-KK)^k, \end{aligned} \tag{D.2}$$

or, in a more convenient form

$$\begin{aligned} \delta B_{2n} = & -2n \int_0^1 dt \varepsilon \delta K e (R - KK + t^2 ee)^{n-1} \\ & +2n \int_0^1 dt \varepsilon (\delta K e - K \delta e) (R - t^2 KK + t^2 e^2)^{n-1}, \end{aligned} \tag{D.3}$$

that is particularly useful to impose the asymptotic conditions for AAdS spacetimes.

References

[1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
 [2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].

- [3] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)].
- [4] C. Fefferman and C.R. Graham, *Conformal Invariants*, in *The mathematical heritage of Elie Cartan (Lyon 1984)*, Astérisque, 1985, Numero Hors Serie, 95.
- [5] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [[hep-th/0002230](#)].
- [6] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev.* **D 15** (1977) 2752.
- [7] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)].
- [8] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 104001 [[hep-th/9903238](#)].
- [9] J.D. Brown and J. York, James W., *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev.* **D 47** (1993) 1407.
- [10] S. Das and R.B. Mann, *Conserved quantities in Kerr-anti-de Sitter spacetimes in various dimensions*, *JHEP* **08** (2000) 033 [[hep-th/0008028](#)].
- [11] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, [hep-th/0404176](#).
- [12] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [[hep-th/0505190](#)].
- [13] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for gravity with locally AdS asymptotics*, *Phys. Rev. Lett.* **84** (2000) 1647 [[gr-qc/9909015](#)].
- [14] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for even dimensional asymptotically AdS gravity theories*, *Phys. Rev.* **D 62** (2000) 044002 [[hep-th/9912045](#)].
- [15] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **06** (2005) 023 [[hep-th/0504233](#)].
- [16] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Vacuum energy in odd-dimensional AdS gravity*, [hep-th/0412046](#).
- [17] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite action principle for Chern-Simons AdS gravity*, *JHEP* **06** (2004) 036 [[hep-th/0405267](#)].
- [18] O. Miskovic and R. Olea, *On boundary conditions in three-dimensional AdS gravity*, *Phys. Lett.* **B 640** (2006) 101 [[hep-th/0603092](#)].
- [19] R.C. Myers, *Higher derivative gravity, surface terms and string theory*, *Phys. Rev.* **D 36** (1987) 392.
- [20] F. Müeller-Hoissen, *Gravity actions, boundary terms and second order field equations*, *Nucl. Phys.* **B 337** (1990) 709.
- [21] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, gauge theories and differential geometry*, *Phys. Rept.* **66** (1980) 213.

- [22] M. Spivak, *Differential geometry, a comprehensive introduction*, Publish or Perish (1979).
- [23] Y. Choquet-Bruhat and C. Dewitt-Morette, *Analysis, manifolds and physics*, North Holland (2001).
- [24] G. Kofinas and R. Olea, *Vacuum energy in Einstein-Gauss-Bonnet AdS gravity*, *Phys. Rev. D* **74** (2006) 084035 [[hep-th/0606253](#)].
- [25] G. Kofinas and R. Olea, in preparation.
- [26] J. Katz, *A note on Komar's anomalous factor*, *Class. Quant. Grav.* **2** (1985) 423;
J. Katz, J. Bicak and D. Lynden-Bell, *Relativistic conservation laws and integral constraints for large cosmological perturbations*, *Phys. Rev. D* **55** (1997) 5957 [[gr-qc/0504041](#)].
- [27] M. Henneaux and C. Teitelboim, *Asymptotically anti-de Sitter spaces*, *Commun. Math. Phys.* **98** (1985) 391.
- [28] S.W. Hawking, C.J. Hunter and M.M. Taylor-Robinson, *Rotation and the AdS/CFT correspondence*, *Phys. Rev. D* **59** (1999) 064005 [[hep-th/9811056](#)].
- [29] A.M. Awad and C.V. Johnson, *Higher dimensional Kerr-AdS black holes and the AdS/CFT correspondence*, *Phys. Rev. D* **63** (2001) 124023 [[hep-th/0008211](#)].
- [30] G.W. Gibbons, M.J. Perry and C.N. Pope, *The first law of thermodynamics for Kerr-anti-de Sitter black holes*, *Class. and Quant. Grav.* **22** (2005) 1503 [[hep-th/0408217](#)].
- [31] R. Clarkson and R.B. Mann, *Eguchi-Hanson solitons in odd dimensions*, *Class. and Quant. Grav.* **23** (2006) 1507 [[hep-th/0508200](#)].
- [32] R. Clarkson and R.B. Mann, *Soliton solutions to the Einstein equations in five dimensions*, *Phys. Rev. Lett.* **96** (2006) 051104.
- [33] T. Eguchi and A.J. Hanson, *Asymptotically flat selfdual solutions to euclidean gravity*, *Phys. Lett. B* **74** (1978) 249.
- [34] D. Astefanesei, R.B. Mann and C. Stelea, *Nuttier bubbles*, *JHEP* **01** (2006) 043 [[hep-th/0508162](#)].
- [35] H. Cebeci, O. Sarioglu and B. Tekin, *Negative mass solitons in gravity*, *Phys. Rev. D* **73** (2006) 064020 [[hep-th/0602117](#)].
- [36] L.F. Abbott and S. Deser, *Stability of gravity with a cosmological constant*, *Nucl. Phys. B* **195** (1982) 76.
- [37] S. Deser and B. Tekin, *Gravitational energy in quadratic curvature gravities*, *Phys. Rev. Lett.* **89** (2002) 101101 [[hep-th/0205318](#)].
- [38] S. Deser and B. Tekin, *Energy in generic higher curvature gravity theories*, *Phys. Rev. D* **67** (2003) 084009 [[hep-th/0212292](#)].
- [39] N. Deruelle and J. Katz, *On the mass of a Kerr-anti-de Sitter spacetime in D dimensions*, *Class. and Quant. Grav.* **22** (2005) 421 [[gr-qc/0410135](#)].
- [40] G. Barnich and G. Compere, *Generalized smarr relation for Kerr AdS black holes from improved surface integrals*, *Phys. Rev. D* **71** (2005) 044016 [*Erratum ibid.* **D 71** (2006) 029904] [[gr-qc/0412029](#)].
- [41] S. Deser, I. Kanik and B. Tekin, *Conserved charges of higher D Kerr-AdS spacetimes*, *Class. and Quant. Grav.* **22** (2005) 3383 [[gr-qc/0506057](#)].

- [42] G.W. Gibbons, M.J. Perry and C.N. Pope, *AdS/CFT Casimir energy for rotating black holes*, *Phys. Rev. Lett.* **95** (2005) 231601 [[hep-th/0507034](#)].
- [43] R.M. Wald, *Black hole entropy in the Noether charge*, *Phys. Rev. D* **48** (1993) 3427 [[gr-qc/9307038](#)].
- [44] K. Skenderis, *Asymptotically anti-de Sitter spacetimes and their stress energy tensor*, *Int. J. Mod. Phys. A* **16** (2001) 740 [[hep-th/0010138](#)].
- [45] S. Nojiri and S.D. Odintsov, *Conformal anomaly for dilaton coupled theories from AdS/CFT correspondence*, *Phys. Lett. B* **444** (1998) 92 [[hep-th/9810008](#)].
- [46] S. Nojiri and S.D. Odintsov, *Is brane cosmology predictable?*, *Gen. Rel. Grav.* **37** (2005) 1419 [[hep-th/0409244](#)].
- [47] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Transgression forms and extensions of Chern-Simons gauge theories*, *JHEP* **02** (2006) 067 [[hep-th/0601081](#)].
- [48] R. Stora, *Algebraic structure of chiral anomalies*, in *Recent progress in gauge theories*, H. Lehmann ed., NATO ASI Series, Plenum, NY (1984).
- [49] B. Zumino, *Chiral anomalies and differential geometry*, in *Relativity, groups and topology II*, B.S. De Witt and R. Stora eds., North Holland, Amsterdam (1984).
- [50] J. Mañes, R. Stora and B. Zumino, *Algebraic study of Chiral anomalies*, *Commun. Math. Phys.* **102** (1985) 157.
- [51] L. Alvarez-Gaumé and P. Ginsparg, *The structure of gauge and gravitational anomalies*, *Ann. of Phys.* **161** (1985) 423.
- [52] M. Bañados, R. Olea and S. Theisen, *Counterterms and dual holographic anomalies in CS gravity*, *JHEP* **10** (2005) 067 [[hep-th/0509179](#)].
- [53] P. Ramond, *Field Theory, A Modern Primer*, Addison-Wesley, Pub. Co., Menlo Park (1980).
- [54] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [[gr-qc/9403028](#)].
- [55] S. Hollands, A. Ishibashi and D. Marolf, *Comparison between various notions of conserved charges in asymptotically AdS-spacetimes*, *Class. and Quant. Grav.* **22** (2005) 2881 [[hep-th/0503045](#)].
- [56] S. Hollands, A. Ishibashi and D. Marolf, *Counter-term charges generate bulk symmetries*, *Phys. Rev. D* **72** (2005) 104025 [[hep-th/0503105](#)].